MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXXIV. MARCH 14, 1924. NO. 5

Dr J. L. E. DREYER, PRESIDENT, in the Chair.

Clarence Augustus Chant, Ph.D., Professor of Astrophysics, University of Toronto;
George Selwyn Clark-Maxwell, St. Leonard's Rectory, Bridgnorth; and
Martin Christopher Johnson, B.A., Physics Department, University of Birmingham,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Fedor Ivanovitsch Blumbach, Chief Meteorologist of the Central Chamber of Weights and Measures, Leningrad, Russia (proposed by F. Robbins);
Mary Ackworth Evershed, Highbroom, Ewhurst, near Guildford, Surrey (proposed by J. Evershed);
Alexei Nikolaievitsch Kriloff, Member of the Russian Academy of Sciences, National Hotel, Upper Bedford Place, London, W.C. 1 (proposed by F. Robbins);
Clayton Conyers Morrell, M.D.Brux., L.R.C.S., L.R.C.P. Edin., D.P.H.Lond., 28 Stratford Place, Camden Square, London, N.W. 1 (proposed by R. Conyers Morrell); and
William R. White, City Engineer's Office, Bloemfontein, O.F.S., S. Africa (proposed by W. J. Hussey).

One hundred and twenty-four presents were announced as having been received since the last meeting, including, amongst others:—

Cape of Good Hope, Astrographic Catalogue, vol. 10, zone $-50°$.
Munich Observatory, Neue Annalen, Band V.

Paris Observatory, Carte photographique du Ciel, zones +18°, +20°, +22° (21 charts).
J. H. Jeans, Report on Radiation and the Quantum Theory. Second Edition.
R. T. Gunther, Set of Plates illustrating "Early Science in Oxford."
J. Evershed, 14 slides of prominences, spectra, etc.
Mount Wilson Observatory, 5 positives on film of dark nebulæ, etc.

On the Relation between the Masses and Luminosities of the Stars.
By A. S. Eddington, M.A., M.Sc., F.R.S. (Plate 8.)

1. A theory of the stellar absorption-coefficient should, if successful, lead to formulæ determining the absolute magnitude of any giant star of which the mass and effective temperature are known. I have hitherto laid most stress on whether the theory will predict the absolute magnitude of Capella. The present position of that problem was summarised in my last paper;* although there appears to have been some measure of success, the final conclusion is not yet certain.

In this paper we shall consider the differential instead of the absolute results of the theory. We are not yet certain what should be the form of the absolute factor occurring in the formula connecting total radiation and mass; but apart from this factor, the form of the law seems to be fixed within narrow limits. Instead of constructing the absolute factor from physical constants we shall be content to determine its value from the observational data for Capella; and then it ought to be possible to calculate the luminosity of any other giant star, the result depending differentially on Capella.

Using the constant determined from Capella, we shall find that the formulæ of the theory appear to predict correctly the absolute magnitudes of all other ordinary stars available for the test, *regardless of whether they are giants or dwarfs.*

The evidence for this statement is shown graphically in fig. 1.

According to the giant and dwarf theory the absolute magnitude is a double-valued function of mass and effective temperature; thus a star of mass 1 and temperature 5860° has two possible magnitudes: (1) that of the Sun at present; (2) that of the Sun when it passed through the same temperature on the upgrade with a much larger surface area than now. It is the latter magnitude that the theory attempts to predict; but the former magnitude is actually situated on the theoretical curve.

If the theory gives the right magnitudes of the wrong stars, it is presumably wrong; if so, the question of its absolute agreement for Capella becomes of minor importance. But it would be surprising if the accordance shown in fig. 1 arose from mere accident, and we must face the question whether the stars there shown are really the "wrong" stars.

The suggestion is that even the dense stars like the sun are in the condition of a perfect gas, and will rise in temperature if they contract.

* *M.N.*, **84**, 104.

In short, all ordinary stars are "giants" according to the usual implication of the term. In the course of this paper theoretical reasons will be given for believing that under stellar conditions matter should be able to contract to an enormously high density before deviations from the laws of a perfect gas become appreciable.

The present results come into conflict with the Lane-Ritter theory of stellar evolution as incorporated in the giant and dwarf theory at present almost universally accepted. Strong initial opposition to the results in this paper will doubtless be felt on that account; a discussion of the nature and extent of the conflict is given in § 12.

Two recent investigations * have appeared containing diagrams which may be compared with fig. 1. Both indicate that when absolute magnitude is plotted against mass the results tend to fall into a continuous line—a conclusion difficult to reconcile with the giant and dwarf theory. The diagram of Russell, Adams, and Joy embodies a great quantity of statistical data and gives mainly independent support to fig. 1. Hertzsprung's results are not independent, since his data have been used by me *en bloc*, along with other evidence; owing to various details of the treatment (use of bolometric instead of visual magnitude, reduction to uniform effective temperature, determination of the mass-ratio of components of double stars) there are differences between the diagrams, and it will be seen that the scatter of his points has become considerably reduced.

2. The determination of the absolute magnitude of a gaseous star of given mass, according to the theory of radiative equilibrium, involves the calculation of an auxiliary quantity β from the equation

$$1 - \beta = 0{\cdot}00309\ M^2\mu^4\beta^4 \quad . \quad . \quad . \quad . \quad (1)$$

where M is the mass in terms of the sun's mass, and μ the average molecular weight. If account is taken of the slight increase of μ from the centre outwards, the correct value to adopt is that corresponding to a point where the temperature is $\frac{2}{3}$ of the central temperature.†

The total radiation of the star is proportional to $M(1-\beta)/k$ where k is the coefficient of absorption. In my earlier papers ‡ k was believed to be approximately independent of temperature and density, so that the total radiation was considered to be proportional to $M(1-\beta)$. Absolute magnitudes calculated on this basis have been used in several researches.

On introducing the modern conception that absorption is an essentially discontinuous process, it was found that k could be more directly calculated as an emission-coefficient. All theories agree that the main part of the variation of k with temperature and density is given by $k \propto \rho/\mu T^3$. This leads to the result that the total radiation is proportional to $M(1-\beta)^2/\beta$, giving a considerably more rapid change of luminosity with mass.§

Detailed theories of emission supplement this by an additional factor of variation of k with temperature; but the additional factor introduces only a small correction in practical applications, and the possible doubt as to its precise form is not a serious hindrance. For

* E. Hertzsprung, *Bull. Ast. Inst. Netherlands*, No. 43. Russell, Adams, and Joy, *Pub. Ast. Soc. Pac.*, 35, 189.

† See § 13. ‡ *M.N.*, 77, 16, 596. § *Zeits. für Physik.*, 7, 389.

the theory of nuclear capture the factor is $T^{-\frac{1}{2}}$; for Kramers' theory of capture it is the same, together with a logarithmic term of which the variations are usually less important. For capture of electrons by radiation under acceleration according to the classical theory, the factor is $T^{-0\cdot9}$; and for the emission at encounters of negative electrons there is no additional factor.* The last two are examples of theories proved to be untenable, but they illustrate the difficulty of disturbing seriously the law proposed. We shall accept the additional factor $T^{-\frac{1}{2}}$, so that

$$k \propto \rho/\mu T^{\frac{7}{2}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

According to E. A. Milne† this gives the right order of magnitude of the optical absorption in the outer layers of the star—an extrapolation far beyond the range here required.

Introducing the effective temperature T_e, this leads to the following formula for the total radiation L.‡

$$L = \text{const.} \times M^{\frac{7}{2}}(1-\beta)^{\frac{3}{2}}\mu^{\frac{1}{2}}T_e^{\frac{1}{2}} \quad . \quad . \quad . \quad (3)$$

The constant will be determined from the observational data for Capella. I have shown in the previous paper that μ is likely to be nearly 2·2 for Capella, and there will not be much change in its value for any ordinary star.§ In the following calculations the value $\mu = 2\cdot11$ is used.

From (1) and (3) we obtain Table I.

TABLE I.

Mass and Absolute Magnitude ($\mu = 2\cdot11$, $T_e = 5200°$).

$1-\beta$.	M.	m.	$1-\beta$.	M.	m.	$1-\beta$.	M.	m.
·001	·1284	14·143	·04	·879	5·211	·26	3·774	−0·052
·0015	·1574	13·173	·05	1·004	4·645	·28	4·137	−0·312
·002	·1820	12·484	·06	1·123	4·178	·30	4·529	−0·562
·0025	·2036	11·950	·07	1·240	3·777	·35	5·675	−1·156
·003	·2233	11·513	·08	1·354	3·426	·40	7·117	−1·718
·004	·2583	10·823	·09	1·468	3·111	·45	8·984	−2·264
·005	·2895	10·286	·10	1·582	2·825	·50	11·46	−2·805
·006	·3176	9·848	·12	1·812	2·322	·55	14·84	−3·354
·008	·3683	9·154	·14	2·050	1·884	·60	19·62	−3·919
·010	·4135	8·615	·16	2·297	1·494	·65	26·66	−4·516
·015	·5117	7·632	·18	2·557	1·138	·70	37·67	−5·162
·02	·5968	6·929	·20	2·831	0·812	·75	56·15	−5·882
·025	·6739	6·381	·22	3·124	0·507	·80	90·63	−6·714
·03	·746	5·929	·24	3·437	0·220			

Add to m the temperature-term, $-2\log_{10}(T_e/5200)$.

* *M.N.*, **83**, 40, 44; **84**, 118. † *Phil. Mag.*, **47**, 226. ‡ *M.N.*, **83**, 107.

§ A possible refinement taking account of the slight variation of μ from star to star is considered in § 11.

Digitized by the Internet Archive
in 2026

https://archive.org/details/monthly-notices-of-the-royal-astronomical-society_1924-03-14_84_5

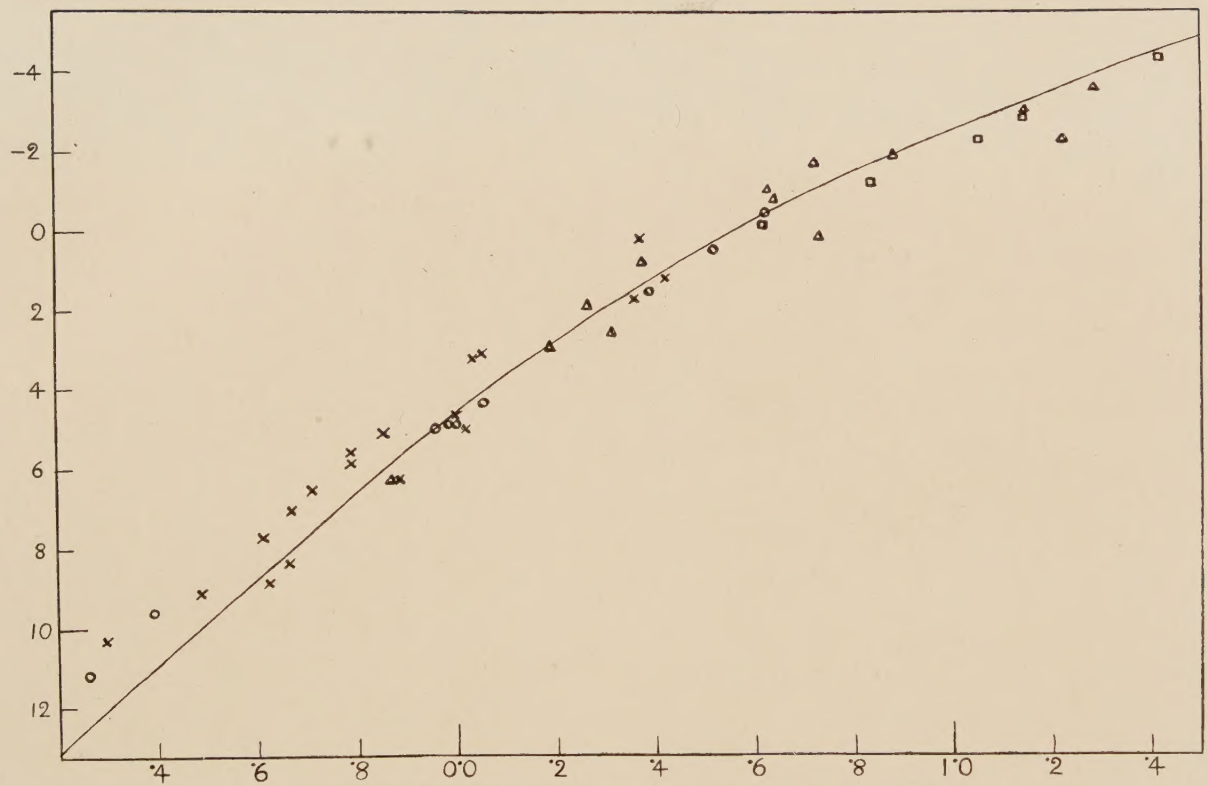


Fig. 1.—ORDINATES—ABSOLUTE MAGNITUDE. ABSCISSAE—LOG. MASS
O First Class. X Second Class. □ Cepheids. Δ Eclipsing Variables.

The table gives the absolute bolometric magnitude m of a gaseous star of mass M, and the same effective temperature as Capella, say 5200°. For other temperatures apply the correction stated at the foot of the table corresponding to the term $T_e^{\frac{1}{2}}$ in (3).

Throughout the paper I have tried to take account of the fact that the same spectral type indicates lower effective temperatures in the more diffuse stars than in the dense stars. At present the amount of difference can only be roughly guessed; but the allowance of 12 per cent. between Capella and the sun is, I think, not excessive. As the assignment of effective temperatures to the stars treated in this paper involves a certain amount of arbitrary judgment, it is satisfactory to note that the whole change from 3000° to 12000° only introduces a difference of $1^m{\cdot}2$. The adopted effective temperature, however, also determines the reduction from visual to bolometric magnitude.* For the hotter stars the two corrections more or less cancel one another; and it is chiefly for temperatures between 3000° and 4000°, where they act in the same direction, that an accurate knowledge of the temperature is important. It is important also in the treatment of Cepheids and eclipsing variables where the data are obtained by more indirect methods.

The numbers given in Table I. are plotted as a curve in fig. 1, the ordinates being absolute magnitude, and the abscissæ logarithm of the mass. The diagram includes nearly the whole observed range of stellar mass and magnitude. The results obtained from observation are represented by circles, crosses, squares, and triangles, according to the nature of the data; full details are given in Tables II.–V., *infra*.

The theoretical curve is for temperature 5200°, and would have to be shifted a little up or down before comparison could be made with a star of different temperature. In order to make the comparison with a fixed curve the observed points have been displaced to the corresponding extent, *i.e.* the observed magnitudes shown in fig. 1 have been reduced to 5200°. The vertical distance of the point from the curve thus represents the actual residual O – C given in the tables of results which follow.

The only serious indication of a systematic deviation of the observed points from the theoretical curve is at the extreme left of the diagram, where the curve appears to run rather too low. According to the discussion in § 11, this divergence will disappear when a closer approximation is made in determining the theoretical curve.

The equation of the theoretical curve here shown contains only two disposable constants, viz. the empirical constant given by the observed mass and luminosity of Capella, and the mean molecular weight derived from the theory of ionisation in the interior of a star.

3. *First-class Determinations.*—The number of stars of reasonably well-determined mass and absolute magnitude is very limited. In general we require (1) a good double star orbit, (2) a large parallax, (3) the ratio of the masses of the two components. The best data are for Capella, Sirius, α Centauri, and the Sun.

Krueger 60, which has been very carefully studied, could be added to the list were it not that the third requirement is lacking. Determina-

* For this a curve based on *Monthly Notices*, 77, 605, Table III., has been used.

tions of the ratio of the masses range from 6 : 5 to 3 : 1, and no reliance can be placed on them. Thus we know the total mass of the system, but not the individual masses of the components. In these circumstances it seems best to divide the total mass in such a way that the calculated difference of brightness agrees with the observed difference. Both components then stand off from the curve to the same extent and furnish only one independent residual to check the theory; but by this method we obtain one good test of the curve instead of two very inaccurate tests.*

Table II. shows the particulars of these stars.

TABLE II.

First-class Stars.

Star.	Type.	T_e.	Parallax.	Mass.	m (vis.).	m (bol.)	m (calc.)	O – C.
Capella, *b*	G0	5200	″·063	4·18	– 0·26	– 0·36	– 0·34	– ·02
,, *f*	F0	7400	...	3·32	0·24	0·22	0·02	+ ·20
Sirius	A0	10500	·373	2·45	1·28	0·97	0·67	+ ·30
α Centauri, *b*	G5	5000	·748	1·14	4·70	4·53	4·16	+ ·37
,, *f*	K5	3700	...	0·97	6·07	5·24	5·09	+ ·15
Sun	G0	5860	...	1·00	4·9	4·89	4·56	+ ·33
Krueger 60, *b*	Ma	3200	·260	0·248	11·51	10·08	11·49	– 1·41
,, *f*	Ma	3200	...	0·182	13·01	11·58	12·91	– 1·33

The main result is contained in the last three columns. The column m (bol.) gives the absolute bolometric magnitude derived by applying the corrections in *M.N.*, **77**, 605, Table III., to the absolute visual magnitude determined from observation; m (calc.) is deduced from the mass by Table I., the correction for effective temperature being included. The residuals, O – C, measure the agreement of theory and observation.

When the writer first made the calculations of Table II. it was not with any expectation that the residuals would be small. According to the view then held, the residuals would measure the reduction in brightness of these dense stars as compared with diffuse stars of the same mass and effective temperature. Current estimates suggested that the residuals would be about $+1^m$ for Sirius, $+4^m$ for the Sun and α Centauri, and at least $+8^m$ for Krueger 60. The smallness of the residuals suggests that for all these stars the deviations from the laws of a perfect gas are inappreciable. The theoretical reason for this behaviour will be considered in § 9.

Before passing on to secondary evidence we may note the general reasonableness of the predictions of Table I. even at the two extremes. We can see that the theoretical formula does not run off badly at the

* Since the two components of a double star sometimes fall at widely separated parts of the curve, I have indicated them both in fig. 1, although (when the method of Krueger 60 is used) they furnish only one independent test. In interpreting fig. 1 it should be remembered that for this reason some of the residuals appear in duplicate.

extreme limits, as a mere interpolation formula is likely to do. The faintest known stars are of absolute magnitude 13 to 15 visual, or about 11 to 13 bolometric. This, according to the table, indicates about $\frac{1}{4}$ to $\frac{1}{6}$ of the sun's mass, which is presumably of the right order. The greatest masses ordinarily found in spectroscopic binaries are from 10 to 20, giving a theoretical absolute magnitude $-2{\cdot}5$ to -4; this agrees with our usual impression as to highest magnitudes ordinarily reached. Even Plaskett's exceptionally massive star would have magnitude about $-6{\cdot}5$, which does not seem unreasonable.

4. *Second-class Determinations.*—The results for the remaining stars for which tolerably accurate data are known are collected in Table III. The accuracy must not be expected to reach the standard of Table II. An error of 10 per cent. in the parallax gives an error of 30 per cent. in the mass, leading to residuals ranging from about $\frac{1}{2}$ to $1\frac{1}{4}$ magnitudes according to the mass of the star, the lightest stars being most affected.

TABLE III.

Second-class Stars.

Star.	Type.	T_e.	Parallax.	Mass.	m (vis.).	m (bol.).	m (calc.).	O−C.
ϵ Hydræ, *b*	F9	5500	·0206	3·64	0·27	0·22	0·00	+0·22
,, *f*	...	5500	...	2·29	1·77	1·72	1·46	+0·26
β Aurigæ, *b*	A0*p*	10500	·025	2·38	−0·19	−0·50	0·77	−1·27
,, *f*	A0*p*	10500	...	2·34	−0·19	−0·50	0·82	−1·32
Procyon	F5	6800	·308	1·13	2·92	2·92	3·92	−1·00
δ Equulei, *b*	F5	6800	·057	1·01	4·43	4·43	4·39	+0·04
,, *f*	F5	6800	...	1·00	4·53	4·53	4·43	+0·10
η Cassiop., *b*	F8	6200	·184	0·72	4·99	4·99	5·93	−0·94
,, *f*	K5	3800	...	0·41	8·73	7·99	8·93	−0·94
ζ Herculis, *b*	G0	5700	·109	1·09	3·23	3·20	4·22	−1·02
,, *f*	...	5000	...	0·51	6·73	6·56	7·65	−1·09
70 Ophiuchi, *b*	K0	4400	·189	1·05	5·66	5·27	4·61	+0·66
,, *f*	K4	3900	...	0·77	7·36	6·71	6·04	+0·67
ξ Boötis, *b*	G6	4900	·164	0·62	5·87	5·69	6·81	−1·12
,, *f*	K4	3900	...	0·47	7·89	7·24	8·27	−1·03
85 Pegasi, *b*	G0	5800	·095	0·62	5·75	5·73	6·66	−0·93
,, *f*	...	3200	...	0·31	10·95	9·52	10·38	−0·86
μ Herculis, *b*	M0	3100	·110	0·46	10·42	8·86	8·57	+0·29
,, *f*	M0	3100	...	0·42	10·92	9·36	8·99	+0·37
o_2 Eridani, *b*	B9	11000	·202	0·21	11·27	10·88	11·15	(−0·27
,, *f*	M0	2900	...	0·20	12·67	10·81	12·53	−1·72

The following remarks on the nature and sources of the data apply to both Tables II. and III.

The principal source is a table by E. Hertzsprung.* The parallax, combined mass, and absolute magnitudes are always taken directly from his list.† I sometimes felt inclined to differ from his judgment, but decided that the test would be more satisfactory if I did not allow myself any freedom of choice. The bracket in the first column indicates that the ratio of the masses was obtained by the method described for Krueger 60, so that the two residuals are not independent. The method was not applicable to Procyon or o_2 Eridani; for the former the determination of the mass-ratio is probably good, but for the latter it cannot be very trustworthy, and I should prefer to reject the star.

The companions of Sirius and Procyon are unsuitable for the test, for the reasons explained in § 10. The brighter component of o_2 Eridani should be rejected for similar reasons although it agrees with the curve; the result in Table III. is only given for reference, and is not represented in fig. 1. With these three exceptions all Hertzsprung's stars have been used; 80 Tauri is included in a fuller discussion of the Hyades which follows. The stars δ Equulei and ϵ Hydræ are additional to Hertzsprung's list, the parallax being obtained by comparing spectroscopic and visual orbits.‡ In the case of ϵ Hydræ the spectroscopic orbit is that of the bright component, whereas the visual orbit is the relative orbit. The mass-ratio is thus required in deducing the parallax and combined mass. For this reason the total mass is found to be much higher than Aitken's value 3·3, which appears to be based on the assumption that the masses are nearly equal; there is no evidence for this assumption, which was presumably made *faute de mieux*.

Perhaps it may be thought that Procyon deserves a place among the first-class determinations. The parallax and mass-ratio are probably trustworthy; but the orbit is uncertain, being based on meridian observations supplemented by a very small arc of micrometric measures. Recent measures do not agree well with the adopted orbit, and it is likely that the inclination is in error. According to theory the mass of the bright component should be 1·5, and of the combined system 2·0; this is within the margin of uncertainty.

The parallax of β Aurigæ is derived on the very probable assumption that it belongs to the Ursa Major stream; but since it is near the apex of the stream the result may be subject to large errors, and the absolute magnitude is uncertain. The mass is trustworthy, being obtained independently from spectroscopic observations. A presumably better determination for β Aurigæ is obtained in § 7. Both results are represented in fig. 1.

The spectral types of many of these stars were taken from *L.O.B.*, No. 343. The effective temperatures were inferred from the types, with some allowance for the influence of the density of the star on the spectrum.

* *B.A.N.*, No. 43.

† Except the magnitude of the sun, for which Hertzsprung gives 4·67; this would bring it still closer to the curve.

‡ Aitken, *The Binary Stars*, p. 209.

5. *The Hyades.*—There are 6 double stars of known period in the Hyades * all of much the same brightness. Although the orbits are not very accurate individually, the mean will afford a good comparison with the theory. Their mean parallax is known with considerable accuracy from the geometry of the moving cluster, viz. 0″·027.

To avoid difficulties in averaging it appears best to invert our previous procedure, and determine by Table I. theoretical masses corresponding to the observed absolute magnitudes. The results are—

Burnham G.C.	M_1.	M_2.	M_1+M_2.
2134	1·08	0·72	1·80
2154	1·03	0·66	1·69
2187	0·83	0·73	1·56
2230	1·48	0·81	2·29
2381	0·93	0·93	1·85
2383	1·16	0·58	1·75

The predicted mean mass of a system is thus 1·82. Hertzsprung found for the mean hypothetical parallax (for mass 2) the value 0″·026, which corresponds to a mean mass 1·79. If we average the masses instead of the hypothetical parallaxes of the individual systems, the mean mass is 1·93; in either case the agreement with the prediction is very good.

This has been represented as a first-class determination in fig. 1, at the point corresponding to the mean mass of the 12 components.

6. *Cepheid Variables.*—The Cepheid variables can be used for testing the curve if the pulsation theory is accepted.† The method is as follows:—

Assume for trial an arbitrary value of the mass with the absolute magnitude corresponding to it by Table I. From the effective temperature (inferred from the observed spectral type) and the absolute magnitude the radius can be deduced. From the mass and radius the mean density is obtained. The central density is 54·25 times greater.‡ The period P is then obtained by the equation

$$P\sqrt{\rho_c} = 0{\cdot}29(\gamma\alpha)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (4)$$

where $(\gamma\alpha)^{\frac{1}{2}}$ is taken from *M.N.* **79**, 15, Table V., the argument $1-\beta$ being given by Table I. of the present paper.

Thus to each point of the theoretical curve will correspond a theoretical period; and we must repeat the process until we find the point which gives the period actually observed. This will give us a *calculated*

* E. Hertzsprung, *B.A.N.*, No. 16.

† *M.N.*, **79**, 2, 177.

‡ According to § 13 this factor is somewhat too large, but we must continue to use it here, since it corresponds to the theory underlying equation (4). The period will depend on the mean density rather than on the density at a particular point, so that in (4) ρ_c is to be regarded as a symbol for 54·25 ρ_m, whatever the actual central density may be.

mass and absolute magnitude. The observed absolute magnitude to be compared with the theoretical prediction is taken from Shapley's discussion.* His determination, although somewhat indirect, is entirely independent of the physical theory.

The following results are obtained :—

TABLE IV.

Cepheid Variables.

Star.	Type.	T_e.	Period. d	Mass.	m (vis.).	m (bol.).	m (calc.).	O − C.
Y Ophiuchi	G2	4800	17·113	26·2	−4·0	−4·21	−4·43	+0·22
η Aquilæ	G0	5100	7·176	13·9	−2·62	−2·76	−3·19	+0·43
δ Cephei	F9	5200	5·366	11·3	−2·19	−2·30	−2·77	+0·47
SU Cassiop.	F5	6000	1·950	6·82	−1·2	−1·20	−1·74	+0·54
RR Lyræ	A9	7300	0·567	4·14	−0·35	−0·36	−0·61	+0·25

In the second column Shapley's mean spectral type has been advanced $\frac{3}{10}$ of a type in order to reduce to the system of Adams and Joy, which is probably more homologous with the types used elsewhere in this paper. The column m (vis.) is taken directly from Shapley's paper above cited.

In the course of the calculations it is necessary to assume a value of Γ, the ratio of specific heats of the stellar material. For Table IV. the value for a monatomic gas, viz. $\frac{5}{3}$, has been used. The value depends on the ratio which the translatory energy of the molecules bears to the whole energy of the matter; we have

$$\Gamma = 1 + \frac{2}{3(1 + dI/dE)} \quad . \quad . \quad . \quad . \quad (5)$$

where E is the external (translatory) energy and I the internal energy. Probably 95 per cent. of the molecules in a star are free electrons; since these have no internal degrees of freedom, it seemed natural to assume that the translatory energy must form by far the greater part of the whole. Accordingly dI/dE was neglected.

But a consideration of the change of ionisation of a pulsating star suggests that the true value of Γ is lower. For a typical massive Cepheid the ionisation-level is about 6Å.† If the star is compressed so as to double the temperature this will be reduced to about 3Å. Elements of atomic number near 35 whose L ionisation extends through this particular range may lose 3 or 4 electrons during the compression; elements Nos. 15–20 may lose 1 or 2 K electrons; a great many elements will scarcely be affected. The average loss may perhaps be taken as 1 electron per atom. The energy of ionisation of an electron in these stars is about 7 times the translatory energy of an electron; so that if there are 20 free electrons per atom, dI/dE may be estimated at $\frac{1}{3}$. That

* *Ap. J.*, **48**, 282; *Mount Wilson Contrib.*, No. 153.

† The data are taken from Table VII. *infra.*

is to say, in compressing the star the energy required for increasing the translatory motions of the molecules must be increased 33 per cent. to provide for the additional ionisation. The corresponding value of Γ is 1·5.

The following table shows the effect of making such an allowance for energy of ionisation on the calculations for Y Ophiuchi and η Aquilæ.

Star.	dI/dE.	Γ.	Mass.	m (bol.).	m (calc.).	O − C.
Y Ophiuchi	0·0	1·666	26·23	−4·21	−4·43	+0·22
,,	0·2	1·555	21·67	−4·21	−4·22	+0·01
,,	0·5	1·444	18·21	−4·21	−3·77	−0·44
η Aquilæ	0·0	1·666	13·86	−2·76	−3·19	+0·43
,,	0·2	1·555	12·53	−2·76	−2·98	+0·22
,,	0·5	1·444	10·17	−2·76	−2·53	−0·23

It appears that the correction would probably improve the agreement with observation, or at the worst, lead to residuals of about the same magnitude but on the opposite side.

7. *Eclipsing Variables.*—When spectroscopic orbits have been determined for *both* components of an eclipsing variable, the information is sufficient to furnish another kind of test of the curve. Owing largely to J. S. Plaskett's spectroscopic researches * thirteen eclipsing variables can be used in this manner.

The spectroscopic orbits determine the quantities $M \sin^3 i$ and $a \sin i$, where a is the semi-axis of the relative orbit. The discussion of the light-curve determines i and r/a, where r is the radius of the star. Hence by the combination we find M and r. Inferring the effective temperature from the spectral type, we can compute the absolute bolometric magnitude, since the total radiation is proportional to $r^2 T_e^4$. The formula is

$$m = 4{\cdot}9 - 5 \log_{10}(r/r_s) - 10 \log_{10}(T_e/T_s),$$

r_s and T_s being the radius and effective temperature of the sun.

This result can be compared with the value m (calc.) obtained from the mass by Table I. It may be noted that the visual magnitude does not intervene in this comparison.

The results are collected in Table V. In all cases they refer to the brighter component, which has been identified with the more massive component except in β Lyræ. The values of $\cos i$ and r/a are taken directly from Shapley's investigation,† the "darkened" solution being preferred. When the star is given as ellipsoidal the geometric mean of the three radii is used. The second column refers to the "grade" of the photometric orbit as classified by Shapley, Grade 1 being the most trustworthy.

* *Publications Dominion Astrophysical Obs., Victoria*, 1 and 2.
† *Princeton Contributions*, No. 3, p. 82.

TABLE V.

Eclipsing Variables.

Star.	Gr.	Type.	T_e.	cos *i*	Mass.	*r*/*a*.	*r*(⊙ = 1).	*m*(bol.).	*m*(calc.).	O − C.
V Puppis	1	B1	19000	·274	19·2	·417	7·60	−4·61	−5·00	+0·4
Y Cygni	3	B2	18000	·088	16·6	·166	4·60	−3·29	−4·66	+1·4
β Lyræ	2	B8	12500	·467	13·9	·219	12·09	−3·80	−3·98	+0·2
u Herculis	1	B3	17000	·268	7·6	·290	4·29	−2·89	−2·90	0·0
U Ophiuchi	1	B9	11500	·105	5·36	·240	3·07	−0·47	−1·69	+1·2
Z Vulpec.	2	B3	17000	·013	5·24	·262	3·94	−2·70	−1·98	−0·7
RS Vulpec.	4	A0	10500	·000	4·34	·252	5·16	−1·20	−1·05	−0·2
U Coronæ	2	B3	17000	·148	4·27	·167	2·90	−2·04	−1·43	−0·6
β Aurigæ	1	A0*p*	10500	·228	2·38	·158	2·80	0·13	0·77	−0·6
TX Herculis	3	A2	9800	·065	2·04	·125	1·33	2·04	1·35	+0·7
TV Cassiop.	4	A0	10500	·176	1·83	·272	1·56	1·40	1·68	−0·3
Z Herculis	2	F2	7600	·139	1·56	·117	1·77	2·53	2·55	0·0
W Ursæ Maj.	1	F8*p*	6200	·244	0·74	·323	0·70	5·42	5·81	−0·4

Several of these stars are early B stars, and there is some diversity of opinion as to the effective temperature corresponding to this type. Fowler and Milne suggest 16000° for the reversing layer in B2; the effective temperature would be higher, and I have adopted 18000°. In view of the considerable uncertainty which exists, it is desirable to consider the effect of possible variations of this standard. We have

$$\delta(\mathrm{O}-\mathrm{C}) = -8\delta(\log_{10}\mathrm{T}_e)$$

Thus, for example, taking for V Puppis the temperatures 15000, 19000, 23000, we should obtain the residuals $+1^{m}\cdot2$, $+0^{m}\cdot4$, $-0^{m}\cdot3$, respectively.

There are certain unsatisfactory features in the present discussion of these variables which could only be remedied by a lengthy research. In most cases Shapley's investigation of the light-curves was made before the spectroscopic data were available, so that no attempt has been made to reconcile the results where they are inconsistent. Some examples of this inconsistency may be given. In TV Cassiopeiæ the photometric result is that the faint component contributes only ·077 of the light, which is scarcely reconcilable with the fact that both spectra have been measured. In Z Herculis the bright star is given as having 4·5 times the surface brightness of the faint; but both spectra are classed as F2. In RS Vulpeculæ there seems to be some doubt as to the correct identification of corresponding components in the photometric and spectroscopic orbits. In some cases the components are so close together that the photosphere will be heated above the effective temperature. For these reasons I have not attempted to utilise the

fainter components, which would presumably be affected to a greater extent by uncertainties of this kind.

In these very remote systems the existence of a third comparatively distant component would usually be undetected; and this would give rise to error in the deductions from the light-curve. A third component of Algol is known to exist; but Algol cannot in any case be used for our comparison, since the spectroscopic orbit of the secondary is unknown. The system of β Lyræ presents exceptional difficulties of interpretation, and it is doubtful whether the result in Table V. has any value.

In view of these difficulties we do not feel able to attribute great weight to the individual results of Table V.; but there appears to be no doubt that in a general way they confirm the theoretical mass-luminosity curve. The eclipsing variables supplement the Cepheids in confirming the theory for large masses, the former being high-temperature stars and the latter low-temperature stars.

8. The average of all our residuals is $\pm 0^{m}\cdot 56$, most of which might fairly be attributed to errors of the observational data. Certain refinements of the nature of a second approximation will be required before the definitive theoretical curve is obtained; for example, we have used a constant value of μ throughout. But it would seem that the first approximation is already very nearly correct, except possibly at the extreme left of the curve.

Granting that the gas-laws hold for all ordinary stars, whether dense or diffuse, are we to expect that each star will have the precise luminosity deducible from its mass and effective temperature? In other words, will the theory be accurate individually, or only statistically? It is difficult to see how residual differences could arise, except from abnormal composition or abnormal rotation. As regards composition, an unduly large proportion of hydrogen would make the star fainter; apart from this not much effect is likely to be produced. As regards rotation, E. A. Milne * has found that a rapid rotation makes the star slightly fainter; but the effect is very small until the speed is sufficient to deform the star greatly. I think that what is most to be feared is that peculiar radiating conditions may arise, such that the observed spectrum misleads us as to the true effective temperature; but if this happened it would be a failure of the test rather than of the theory. It may be noted that an unsuspected binary should betray itself by having a magnitude fainter than that predicted from its (combined) mass.

9. *Theoretical Considerations.*—We must now consider whether it is physically likely that a dense star, such as the sun, can obey the laws of a perfect gas.

The failure of the ordinary gas-laws at high densities is due to the finite size of the molecules which behave approximately as rigid spheres with radii of the order 10^{-8} cm. Compression proceeds with increasing difficulty until these spheres are packed tightly; the density is then of the order characteristic of solids and liquids. The idea underlying the giant and dwarf theory is that the maximum density of ordinary matter (say 10–20 gm. per c.c.) is applicable to the stars, and that the devia-

* *M.N.*, 83, 118.

tions from the gas-laws first begin to have serious effect when the density comes within sight of this limit.*

But the atoms in a star are very much smaller than ordinary atoms. Several layers of electrons have been stripped away, and the gas-laws ought therefore to hold up to far greater densities. It appears that in the interior of a star the atoms of moderate atomic weight are stripped down to the K level, and have radii of the order 10^{-10} cm.; lighter elements, such as carbon and oxygen, are reduced to the bare nucleus. The maximum density, corresponding to contact of these reduced atomic spheres, must be at least 100,000, and any star with mean density below 1000 ought to behave as a perfect gas.

It may be asked: Does the removal of outer electrons necessarily reduce the effective size of the atom? Perhaps it is only the boundary-stone, not the boundary, that disappears. The answer seems to be given clearly by physical experiment. An α particle is a helium atom which has lost its "boundary stones," and it appears that it thereby loses its former boundary. It can now enter other atoms, and behaves in every way as a simple charged nucleus with no trace of that resisting boundary which prevents neutral helium gas from being compressed beyond a certain density. It seems clear that the effective size of the atom is determined by the existing peripheral electrons—as we should expect theoretically.

A further question arises as to the effect of the charges of the ions and electrons It seems almost paradoxical that we should be able to force atoms closer together by ionising them, and so making them repel one another. Will not the repulsion of the ion establish a region which other ions are unable to enter, so that the volume of this region constitutes an effective size of the ion? It is very difficult to calculate the effect of these electrical forces; they are not obviously insignificant, at any rate in the stars of small mass. But it is quite easy to see that the effect does not increase when the star contracts; it is just as large when the star is diffuse as when the star is condensed, so that there is no evolution from gaseous to non-gaseous (giant to dwarf) condition.

It has often been pointed out in atomic theory that if inverse-square forces alone are acting no definite scale of size can be obtained. Thus inverse-square electrical forces will not alter the result for inverse-square gravitational forces, viz. that there is no definite scale of size for a giant star of given mass—it is equally comfortable with any radius. Stars of the same mass and different radii form a perfectly homologous series, which can only be disturbed when other than inverse-square forces begin to play an appreciable part. According to current theory this happens when the compression is great enough to bring into importance the inter-atomic forces at impact, which do not follow the inverse-square law. The star then passes into a dwarf equilibrium not homologous with its previous progress. But we have just seen that this change will not occur until the star reaches a density of at least 1000; and electrical forces between the charged atoms and electrons do not lead us to modify

* Since the central density in a giant star is about 20 times greater than the mean density, we expect the deviations to be already appreciable at a mean density of 0·2.

this conclusion, because, being inverse-square forces, they cannot produce a breach of homology. This can be seen in more detail as follows:—

Take any star A in equilibrium, and let B be a precise replica of A with lengths altered in the ratio p and velocities in the ratio $p^{-\frac{1}{2}}$. Hence temperature is altered in the ratio p^{-1} and density in the ratio p^{-3}, giving the familiar law $\rho \propto T^3$. Kinetic energy and potential energy (whether electrical or gravitational) will be altered in the same ratio p^{-1}. Moreover, radiant energy and material energy will be changed in the same ratio, the density of the former being proportional to T^4 and the latter to ρT; these both change in the ratio p^{-4}. Electrical-pressures, gas-pressures, and radiation-pressures are thus all altered in the same ratio, and the replica B will accordingly be in equilibrium.*

To give a simple example, at any moment in a certain star let there be n pairs of positive ions which have approached to a distance r and have a mutual potential energy E. Now let the star contract to 8 times its former density; according to the theory of equilibrium its tempera ture must increase twofold. There will now be n pairs at a distance apart $\frac{1}{2}r$, and the mutual potential energy will thus be 2E. But by the rise of temperature the kinetic energy is also increased twofold, so that the same proportion of potential to kinetic energy is maintained.

Since the least massive stars have the greatest density for given temperature, the effect of the electrical forces, if appreciable at all, should be shown by them. We shall consider Krueger 60 which, according to Table VII. *infra*, has a temperature $2{\cdot}81.10^7$ at a density 360. If it is made of iron, there will be $3{\cdot}8.10^{24}$ atoms per c.c., giving an average separation of $0{\cdot}64.10^{-8}$ cm. The charge of an ion (retaining 3 electrons) will be $23e$, and two such ions will have a mutual potential energy equal to the average kinetic energy of four free molecules when their separation is $0{\cdot}61.10^{-8}$ cm.—practically the average separation. At average separation this potential energy merely cancels the negative potential energy of the ions due to the corresponding free electrons which are at large in the neighbourhood; but if the two ions approach to $0{\cdot}4.10^{-8}$ cm., their mutual energy is increased 50 per cent., whereas there can be no appreciable change in the cancelling term—the average distribution of 46 free electrons giving a practically steady field. Hence the approach can only be made at the expense of their own kinetic energy, which will be just used up if they originally possessed the average velocity. It appears then that two ions cannot on the average approach nearer than $\frac{2}{3}$ their normal distance; effectively, an ion is barred out from $\frac{1}{3}$ of the whole volume.

It does not follow that exclusion by this method has the same effect as exclusion by finite size of the molecules at ordinary temperature;† but we can perhaps estimate the possible order of magnitude of the effect by assuming the analogy to hold. In an ordinary gas if $\frac{1}{3}$ of the whole volume is barred to a molecule, the pressure is increased in the ratio $\frac{6}{5}$, or 20 per cent. It must be remembered, however, that in the present case only $\frac{1}{2^{4}}$ of the whole pressure is due to the ions, the

* The ionisation and the absorption-coefficient involve atomic constants, and are not governed by the same considerations.

† I doubt whether the effect has even the same sign.

remainder being due to the free electrons which are not barred out of these regions.* Thus the increase of pressure is only 1 per cent., and is therefore inappreciable even in the least massive stars.

We scarcely pretend to know all about the inter-atomic forces at present, and it would not surprise me much if some unknown circumstance were to vitiate the argument of this section. But what I wish to bring out is, that in applying the Lane-Ritter theory to stellar evolution we have been influenced by a false analogy between the mutilated stellar atoms and ordinary atoms; we can at least see now that this analogy is unfounded and approach the problem again free from this bias. There is no theoretical reason to expect a change in stellar conditions as the star reaches a density 0·1 – 1; and the indications here found, that the condition of a perfect gas persists up to higher densities, are not to be dismissed as incredible. It would have been more puzzling if we had found that it did not persist.

10. *White Dwarfs.*—The white dwarfs Sirius (*comes*) and o_2 Eridani (bright component) should not be used for our comparisons. They have long presented a difficult problem.

The spectrum of Sirius *comes* is now given as Fo. If this is to be taken literally as indicating an effective temperature of 8000°, it gives in conjunction with the absolute magnitude 11·3 a radius of 19600 km. —much less than Uranus. Determinations of the mass range from 0·75 to 0·95; adopting 0·85, the density is 53000.

According to the views here reached, such a density is not absurd, and we should accept it without demur if the evidence were sufficient. The alternative is that at a very low effective temperature, probably below M*d*, the star is able in some way to produce an imitation of leading features of the F spectrum sufficiently close to satisfy the expert observer. It seems unnecessary to debate these alternatives at length, because, as several writers have pointed out, the question could probably be settled by measuring the Einstein shift of the spectrum, which should amount to about 20 km. per sec. if the high density is correct.

When compared with the curve the companion of Sirius is much too faint for its mass; this would be anticipated on either hypothesis. If the density is really 53000, entirely new considerations enter into the calculation of k, since the electrons are in the capture zone of two or more nuclei simultaneously; moreover, the deviation from the gas-laws may be important at this high density. If the density is moderate, the effective temperature must be in the domain below 3000°, where the visual magnitude diverges steeply from the bolometric

It is rather surprising to find o_2 Eridani, presumably a similar star, but of smaller mass, agreeing quite well with the curve.

The companion of Procyon is also very faint for its mass, but since the spectrum is unknown there is at present no reason to suspect a discrepancy. It may be in the region below M*d* where any degree of visual faintness is possible. I suppose it is only mythological association which suggests the idea that it may be another white dwarf.

* The volume from which the free electrons exclude one another is about $(23)^{-5}$ of the whole.

11. All the calculated absolute magnitudes in the foregoing sections have been based on the adopted value 2·11 for the molecular weight. From (1) and (3) it is deduced that the change Δm in the predicted magnitude due to a small change $\Delta\mu$ in the molecular weight is

$$-\Delta m = \frac{9\beta+8}{4-3\beta}\log_{10}e\cdot\frac{\Delta\mu}{\mu} \quad . \quad . \quad . \quad . \quad (6)$$

Table VI. gives the increase of brightness ($-\Delta m$) for an increase of 0·1 in the molecular weight.

TABLE VI. ($\Delta\mu=0{\cdot}1$.)

$1-\beta$.	Mass.	$-\Delta m$.	$1-\beta$.	Mass.	$-\Delta m$.
0·0	0·0	0·35	0·3	4·5	0·16
0·05	1·0	0·30	0·4	7·1	0·13
0·1	1·6	0·25	0·6	19·6	0·09
0·2	2·8	0·20	0·8	90·6	0·06

Since our calculations are differential, being based on Capella (mass 4·2), a *general* increase of 0·1 in μ would cause a maximum change of $0^m{\cdot}2$, so that an alteration of 2 or 3 tenths would have no very serious effect. But it is more important to inquire whether, having regard to the very wide range of stars investigated, a uniform value of the molecular weight is a reasonably good first approximation.

As a criterion for comparing the degree of ionisation in different stars, we shall use the critical frequency ν_1 introduced in a former paper.* This is defined as follows: If the atoms have an electron at the level corresponding to frequency ν_1, then of those already ionised for all electrons above this level half will have lost this electron also and half will retain it. In general, ν_1 gives the average level down to which the atoms are ionised; but this statement requires some modification when there is much overlapping of successive stages of ionisation, which happens when $h\nu_1/\mathrm{RT}$ is small (or negative). According to equation (12) *loc. cit.*, ν_1 is calculated from

$$\frac{h\nu_1}{\mathrm{RT}} = \log_e\left\{\frac{(2\pi m\mathrm{R})^{\frac{3}{2}}\mathrm{H}}{h^3}\cdot\frac{\mu\mathrm{T}^{\frac{3}{2}}}{\rho}\right\} \quad . \quad . \quad . \quad (7)$$

In place of ν_1 we may use the wave-length λ_1 corresponding to it.

In Table VII. results are given for a representative selection of masses and effective temperatures covering the range of the comparisons previously used. The extent of the table is reduced, because up to mass 2 the effective temperature and mass are closely correlated—an aspect of the well-known correlation of magnitude and spectral type in the dwarf stars. Beyond mass 2 the stars appear to be capable of existing in a wide variety of temperature conditions. The Cepheids and eclipsing variables are similar in mass, but differ greatly in effective temperature; an example of each is accordingly given.

* *M.N.*, **84**, 108.

TABLE VII.

$1-\beta$.	Mass.	T_e.	$10^{-6}T$.	ρ.	$h\nu_1/RT$.	λ_1.	δm.	$\delta' m$.	Example.
·004	·258	3200	28·1	360	1·28	4·0	0·0	−1·6	Krueger 60.
·01	·414	3800	21·3	62	2·60	2·6	+0·3	−0·8	η Cassiop., *f*.
·05	1·00	5800	15·7	4·8	4·71	1·9	+0·5	−0·6	Sun.
·16	2·30	10500	19·2	2·4	5·69	1·3	+0·5	−0·4	Sirius.
·28	4·14	5200	4·2	·012	8·69	3·9	0·0	0·0	Capella.
·60	19·6	5000	2·0	·00033	11·20	6·5	−0·2	+0·2	Cepheid Var.
·60	19·6	18000	15·3	·155	8·10	1·2	+0·2	−0·1	Eclipsing Var.

From the fourth and fifth columns it will be seen how widely these stars differ in physical conditions. They refer to a point in the star where the temperature is $\frac{2}{3}$ of the central temperature, this being the place at which the value of μ is required. I have taken the central density to be 20 times the mean density, as this is probably better than the factor 54·25 hitherto used (see § 13). The next two columns also refer to this point in the star. The columns δm, $\delta' m$, will be explained later. The star mentioned as an "example" satisfies roughly the data of columns 2 and 3.

The essential point of the table is the very small range in the value of λ_1, which is given in Ångstrom units. Between the extreme values 6·5 and 1·2 iron could scarcely lose more than 3 electrons, changing the molecular weight from 2·4 to 2·1; moreover, this refers to massive stars for which (by Table VI.) the consequent difference of magnitude is small. A heavy element, say No. 52, with its L limit in the middle of this range, might lose the whole group of 8 electrons, but the change of molecular weight is then only 0·45, giving a magnitude change of $0^m{\cdot}4$. For less massive stars the greatest possible change of molecular weight is smaller, but it has proportionately more effect on the magnitude, so that the limit of correction to magnitude is about the same.

The column δm gives my estimate of the amount of correction which might be applied to the magnitudes calculated by Table I. in order to allow for the variation of ionisation. Since the object is to indicate the possible uncertainty of the first approximation rather than to advocate a second approximation, I have overestimated the corrections if anything. The table is adequately representative of all stars which have furnished comparisons, but not necessarily of all existing stars. Somewhat larger corrections might be necessary for a star like Betelgeuse of large mass and very low temperature.

If we accept Kramers' theory, the absorption-coefficient k besides containing the factor $\rho/\mu T^{\frac{7}{2}}$ contains an additional factor proportional to $(1+h\nu_1/RT)$. Since the total radiation is inversely proportional to k, the effect of this correcting factor on the absolute magnitude is easily calculated using the values of $h\nu_1/RT$ in column 6. This correction is tabulated as $\delta' m$.

If both corrections δm and $\delta' m$ are applied, the magnitudes for stars of mass $>\frac{1}{2}$ are practically unaltered. For mass $<\frac{1}{2}$ there is a rather rapidly increasing correction amounting to $-0^m{\cdot}5$ for mass 0·4, and to

$-1^m{\cdot}6$ for mass 0·26. Examination of fig. 1 shows that there is some observational support for this second approximation, since the representative points for the stars of small mass lie chiefly above the original curve. In particular, the first-class determination for Krueger 60 is brought into complete agreement.

There are general reasons for accepting a correcting factor varying approximately as $(1+h\nu_1/RT)$ apart from the special postulates of Kramers' theory. The factor represents the ratio of the energy given up on capture at the mean ionisation level to the mean free energy before capture. Such a factor must appear in deducing the emission of radiation from the number of captures.*

The conclusion from this section is that our first approximation, $k \propto \rho/T^{\frac{1}{2}}$ $\mu = \text{const.}$, although at first sight rather crude, is likely to give absolute magnitudes very close to the true values. Further, making the best estimate we can of the effects of a more refined approximation, the slight systematic deviation between theory and observation for the stars of very small mass tends to close up.

12. *Stellar Evolution.*—We have been led to views which in some respects are opposed to the giant and dwarf theory of evolution; and since the latter theory is considered to be strongly supported on most points by observational evidence, it is necessary to make clear the nature and extent of the conflict.

In its observational aspects the giant and dwarf theory is summed up in fig. 2. The leading phenomenon is that the observed distribution of absolute magnitude and spectral type is concentrated along the lines PQ, QR with comparatively small scattering. The evidence for the reality of this dual concentration is extremely strong, and we shall accept it without question. Further, the observational evidence shows that the stars near the line PQ are diffuse with densities comparable to air, and those along QR have densities comparable to water; near the junction the density is of order 0·1. This difference of density is theoretically necessary unless the stars differ enormously in mass.

The discovery of this important statistical distribution appears to be a permanent contribution to our knowledge. We pass on to the evolutionary aspect of the giant and dwarf theory. It is supposed that the line PQR indicates the course of evolution of a single star. If, as usual, it is assumed that the mass of the star remains sensibly constant during its lifetime, this is in conflict with our results; for we have found that the stars at R are essentially less massive than those at P (the difference of magnitude having arisen from this mass-difference) so that no star can pass from P to R.

Considering a star at S, our results give a nearly horizontal line TST′ as its course of evolution. If the energy is derived from contraction, the direction must be from T to T′; since, however, the energy

* In obtaining the formulæ for the theory of nuclear capture, I assumed as an approximation that the mean energy given up on capture was equal to the mean quantum of radiation, and therefore in a constant ratio to the mean free energy before capture. Had I proceeded to a higher approximation, the factor $(1+h\nu_1/RT)$ would naturally have appeared in place of this constant ratio. Milne has, in fact, discussed this correction (*Phil. Mag.*, **47**, 235–239).

is due to some other source, the direction T′T is not definitely excluded but it seems unlikely. In any case S must be a point of quasi-equilibrium, since we seldom if ever find stars on this track except near S. Accordingly QR, instead of being a line of evolution, becomes, according to our view, a locus of equilibrium points—unless we are prepared to admit that the mass of the evolving star changes.

It is in this connection that strong opposition to the conclusions here reached is likely to be felt. The giant and dwarf theory gave an extremely plausible explanation of the distribution PQR, on the view that the turning at Q was due to the density having reached the point

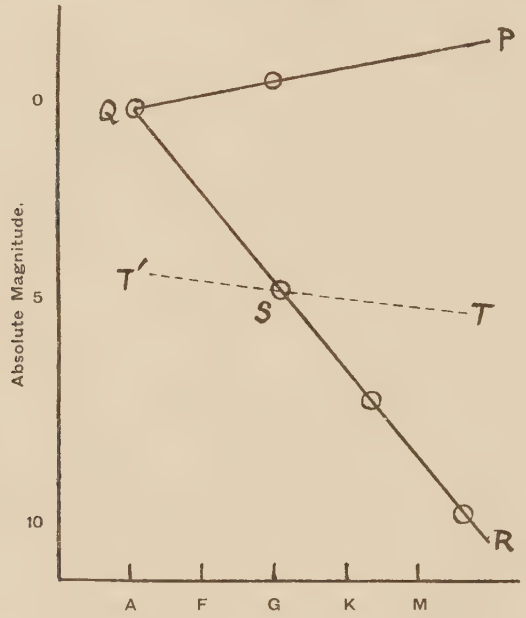


FIG. 2.

at which the star ceases to behave as a gas. We find it necessary to disbelieve this precise explanation, and as regards this particular feature of stellar distribution can offer only vague suggestions in its place. We make the following defence :—

Firstly, the giant and dwarf explanation is essentially bound up with the view that the influence of mass on luminosity is comparatively small ; for otherwise the statistics of stars of systematically different masses cannot outline the evolutionary course of a star of constant mass. It was recognised from the beginning—as a minor detail—that the stars along QR were statistically less massive than along PQ. I suppose that the explanation most generally held was that the lighter stars traversed their course more quickly—although all theoretical arguments tend to show that the heavier stars would reach R first. However that

may be, there is much evidence apart from the present investigation to show that the systematic difference of mass is not a minor detail, and the correction to be applied before PQR can give the course of an individual star is very considerable. When the correction is applied, the "individual" curve PQR will make a much more acute angle than the "statistical" PQR; and the magnitude difference of the ascending and descending states is at any rate greatly exaggerated in the statistical diagram. We have attempted to determine theoretically the actual amount of the correction for mass-difference, and according to our result the corrected angle PQR is not merely more acute but closes up altogether leaving a single line like TT′.

Secondly, we have shown in § 9 that the assumption of the giant and dwarf theory that a deviation from the gas-laws sets in at density 0·1 is based on a false analogy; if it occurs, it requires inter-atomic forces which have not yet appeared in experimental or theoretical physics.

Thirdly, although we can offer no serious attempt at an alternative explanation, it does not seem that explanation is impossible. The difficulty is merely that we have no knowledge of the conditions under which the star's unknown source of energy is liberated; and in these circumstances speculation is unlimited and chiefly unprofitable.

It is possible that a star may gradually diminish in mass during its evolution.* This would happen if it obtains its energy of radiation by annihilating electrons and protons, thus burning itself away. If this suggestion is accepted, our breach with the current giant and dwarf theory is much less wide. A star by losing mass can pass from P to R; and PQR may quite possibly represent the course taken—as the statistical diagram appears to hint. The circles drawn on PQR represent roughly Capella, Sirius, the Sun, η Cassiop. *f*, Krueger 60; these form a series of diminishing mass and (according to Table VII.) of increasing density. The rise of internal temperature may also be continuous.† Thus a star burning itself up may increase continuously in density and *internal* temperature although the *effective* temperature first rises and then falls.

There is therefore no need at present to contradict the current theory in its division of the stars into an ascending and descending sequence (as judged by effective temperature) corresponding to diffuse and dense stars respectively. We have to reject its explanation of this phenomenon; our own explanation cannot be developed in detail owing to ignorance of how the rate of generation of sub-atomic energy depends on temperature and density, and possibly on exhaustion of suitable material.

If the radiant energy of a star arises from annihilation of matter, it may be possible to estimate the relative numbers of stars in different

* In any case the star will lose mass on account of the radiated energy, but we are now going to suppose that the time-scale is so great that this loss is of serious importance.

† There will naturally be parallel courses concentrated near the course of an average star indicated by PQR. Hence the result in Table VII., that the internal temperatures of Sirius and the Sun are out of order, would indicate that they are not quite on the same course.

stages. The abundance of any stage should *ceteris paribus* be proportional to the duration δt of the stage. If δM is the amount of mass carried off as radiant energy during the stage

$$\delta t \propto \int_{M-\delta M}^{M} \frac{dM}{L}.$$

Approximately

$$L \propto M(1-\beta)^2/\beta,$$

neglecting the factor $T^{-\frac{1}{2}}$ in k. Hence

$$dt \propto \frac{\beta}{(1-\beta)^2}\frac{dM}{M} \propto \frac{4-3\beta}{(1-\beta)^3}\,d\beta$$

by (1). Integrating, we have

$$\delta t \propto \delta\left\{\frac{\frac{1}{2}+3(1-\beta)}{(1-\beta)^2}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

From (8) we obtain Table VIII. showing the relative time spent between successive limits of absolute magnitude by a star starting originally with very large mass. The absolute time in years could be calculated quite easily if desired.

TABLE VIII.

Abs. Mag.	Luminosity ($\odot = 1$).	δt.
$< -2{\cdot}5$	> 1000	10
$-2{\cdot}5$ to $0{\cdot}0$	$1000 - 100$	21
$0{\cdot}0$ to $2{\cdot}5$	$100 - 10$	99
$2{\cdot}5$ to $5{\cdot}0$	$10 - 1$	472
$5{\cdot}0$ to $7{\cdot}5$	$1 - 0{\cdot}1$	3670
$7{\cdot}5$ to $10{\cdot}0$	$0{\cdot}1 - 0{\cdot}01$	29800

The last column has a distant resemblance to the frequency of stars between these limits of brightness found in statistical researches. Comparatively few stars will be formed with large initial mass, so that at each descending stage new starters enter the race and the frequency must be multiplied by a continually increasing factor. With plausible assumptions it seems possible to account in this way for the observed frequency distribution; but the possibilities are too elastic to afford any very cogent evidence in favour of the view that the star burns away its mass.

13. This section deals with a point of detail relating to the exactness of the fundamental equation (1). It is not concerned with the main argument of the paper.

In solving the equations of the theory of radiative equilibrium of a gaseous star, I have hitherto made two approximations: (1) that the product ηk is constant; (2) that the mean molecular weight of the material is constant. On Kramers' theory of capture and emission, or

on the nuclear theory, ηk is so nearly constant that the error introduced by assuming μ constant is likely to be the more serious. In the present discussion we shall adhere to the approximation (1) but relax (2), and examine the effect of varying molecular weight.

Since the ionisation decreases with temperature, a slight increase of μ must be expected in passing from the centre to the boundary of the star. We can represent this increase by a law of the form $\mu \propto T^{-s}$. In a typical case with central temperature 10^7 degrees most atoms will be ionised down to the K ring and m may be taken as 2·2. The law $\mu \propto T^{-\frac{1}{2}}$ would give $m = 3{\cdot}3$ (*i.e.* L electrons retained) at $1{\cdot}3 \cdot 10^6$ degrees, and $m = 8{\cdot}8$ (only peripheral electrons ionised) at 10000° near the surface. Probably this would represent the actual conditions closely, but we shall also consider below laws expressing greater and less rates of variation.

We start from the equations *

$$P = \tfrac{1}{3}\alpha T^4/(1-\beta) \qquad P = \mathfrak{R}\rho T/\mu\beta \quad . \quad . \quad . \quad (9)$$

where P is the total pressure (gas-pressure + radiation-pressure). In virtue of the approximation $\eta k = \text{const.}$, β is constant throughout the star. From these

$$P = \kappa\rho^{\frac{4}{3}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\kappa^3 = 3\mathfrak{R}^4(1-\beta)/a\mu^4\beta^4 \quad . \quad . \quad . \quad . \quad (11)$$

But since μ is now considered to be variable, κ is not constant. If, however, we insert in (9)

$$\mu \propto T^{-s} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and then eliminate T, we have

$$P = \kappa'\rho^t \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$t = 4/(3-s)$$

and κ' is constant. If ϕ is the gravitational potential, the hydrostatic equation is

$$\frac{d\phi}{dr} = \frac{1}{\rho}\frac{dP}{dr} = \kappa' t\rho^{t-2}\frac{d\rho}{dr}$$

so that by integration

$$\phi = \kappa'\frac{t}{t-1}\rho^{t-1} \quad . \quad . \quad . \quad . \quad (14)$$

Let $t = 1 + 1/n$, so that

$$n = (3-s)/(1+s) \quad . \quad . \quad . \quad . \quad (15)$$

Then by (14)

$$\rho = \{\phi/\kappa'(n+1)\}^n \quad . \quad . \quad . \quad . \quad (16)$$

Poisson's equation thus becomes

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + \alpha^2\phi^n = 0 \text{ where } \alpha^2 = 4\pi G\{(n+1)\kappa'\}^{-n} \quad . \quad (17)$$

G being the constant of gravitation.

---

* See, for example, *Ap. J.*, **48**, 208.

Let u be the solution of the equation

$$\frac{d^2u}{dz^2}+\frac{2}{z}\frac{du}{dz}+u^n=0$$

which satisfies $u=1$ for $z=0$. Also let $M_n=(z^2du/dz)_{u=0}$. The solutions and the values of M_n for certain values of n have been given by Emden.* Equation (17) will be satisfied by

$$\phi=Uu \qquad r=z/U^{\frac{1}{2}(n-1)}a \quad . \quad . \quad . \quad (18)$$

where U is an arbitrary constant.

If M is the mass of the star

$$GM=\left(r^2\frac{d\phi}{dr}\right)_{\rho=0}=\frac{U^{\frac{1}{2}(3-n)}}{a}\left(z^2\frac{du}{dz}\right)_{u=0}=U^{\frac{1}{2}(3-n)}M_n/a$$

so that

$$\left(\frac{GM}{M_n}\right)^2=\frac{U^{3-n}}{a^2}=\frac{(n+1)^n\kappa'^nU^{3-n}}{4\pi G} \quad . \quad . \quad . \quad (19)$$

By (10), (13), and (16)

$$\frac{\kappa'}{\kappa}=\rho^{\frac{1}{n}-1}=\left\{\frac{\phi}{(n+1)\kappa'}\right\}^{\frac{1}{n}n-1} \quad . \quad . \quad . \quad . \quad (20)$$

Since at the centre $u=1$, ϕ at the centre is U; and for the central value κ_0 of κ, (20) gives

$$\kappa'^n/\kappa_0{}^3=\{U/(n+1)\}^{n-3} \quad . \quad . \quad . \quad . \quad (21)$$

Hence by (19), (21), and (11)

$$\frac{4\pi G^3M^2}{(n+1)^3M_n{}^2}=\kappa_0{}^3=\frac{3\mathfrak{R}^4(1-\beta)}{a\mu_0{}^4\beta^4}$$

so that β is determined by the quartic equation

$$1-\beta=C_n'M^2\mu_0{}^4\beta^4 \quad . \quad . \quad . \quad . \quad (22)$$

where

$$C_n'=4\pi G^3a/3(n+1)^3M_n{}^2\mathfrak{R}^4 \quad . \quad . \quad . \quad (23)$$

Or if M is, as usual, expressed in terms of the sun's mass, the coefficient is

$$C_n=C_n'\odot^2=4\pi G^3\odot^2a/3(n+1)^3M_n{}^2\mathfrak{R}^4 \quad . \quad . \quad (23a)$$

In comparing the results for different laws of variation of molecular weight, it is scarcely fair to consider the same *maximum* value μ_0. Let us then use in all cases the same equation as for constant molecular weight, viz.

$$1-\beta=C_3M^2\mu^4\beta^4,$$

adopting the appropriate average value of μ determined by

$$C_3\mu^4=C_n\mu_0{}^4.$$

* *Gaskugeln*, pp. 78–82.

Our results are then expressed by defining the temperature T_n, to which the required value of μ belongs, in terms of the central temperature T_0, *i.e.*

$$\mu = \mu_0(T_n/T_0)^{-s} \qquad T_n/T_0 = (C_3/C_n)^{1/4s}.$$

The results are contained in Table IX., which extends to laws of variation of μ far beyond what is likely to occur in nature. The values of M_n are taken directly from Emden's tables, except that for $s = \frac{1}{5}$ which is interpolated. The column ρ_c/ρ_m, also taken from Emden, is added for reference; it gives the ratio of the central density to the mean density which is required in some of the astronomical calculations. The line $s = 0$ gives the result for constant molecular weight used hi therto.

TABLE IX.

s	t.	n.	M_n.	C_n.	T_n/T_0.	ρ_c/ρ_m.
$-\frac{3}{11}$	1·22	4·5	1·7357	·00160	·548	6377
$-\frac{1}{6}$	1·25	4	1·8064	·00197	·570	623
0	1·33	3	2·0150	·00309	...	54·25
$\frac{1}{7}$	1·4	2·5	2·2010	·00387	·675	24·08
$\frac{1}{6}$	1·43	2·33	(2·27)	·00421	·679	(20)
$\frac{1}{5}$	1·5	2	2·4107	·00512	·685	11·40
$\frac{2}{5}$	1·66	1·5	2·7176	·00696	·713	6·00
1	2	1	3·1416	·01002	·742	3·29

It will be seen that throughout the whole table the "mean" temperature for which μ is to be calculated when used in equation (1) is between $\frac{1}{2}$ and $\frac{3}{4}$ of the central temperature, and for the more probable laws it is very near to $\frac{2}{3}$.

The only other consequence is that if $s = \frac{1}{5}$ the central density is about 20 times the mean density instead of $54\frac{1}{4}$ times. There is rather less concentration of mass towards the centre; central densities previously given are $2\frac{1}{2}$ times too high, and temperatures $1\frac{1}{3}$ times too high. The change is more apparent than real, since the central temperature and density have been used as a general guide to the conditions to be considered in the interior rather than for the intrinsic interest of this particular point.

14. *Summary.*

1. Assuming on the evidence of previous investigations that the absorption-coefficient is proportional to $\rho/T^{\frac{7}{2}}$, it is possible to calculate the difference of absolute magnitude of any two gaseous (giant) stars of known mass and effective temperature. Hence, using the observed data for Capella, the absolute magnitudes of other stars can be determined differentially.

2. Collecting all suitable data 36 stars furnish comparisons between theory and observation. The average residual is $\pm 0^{m}\cdot 56$, and the maximum discordance is $1^{m}\cdot 7$. The probable errors of the observational data would account for a great part of this difference.

The only stars omitted in the comparison are the two "white dwarfs." For these the internal conditions must (if the observations are not at

fault) be so different from those of a normal star that the theoretical calculations are not expected to apply without modification.

3. More than half the stars used in the comparison are dwarf stars. The agreement of their absolute magnitudes with the predicted magnitudes for gaseous stars is in conflict with the current view that they are too dense to follow the laws of a perfect gas, and that their low luminosity is attributable to deviation from the gas-laws. According to the present results their low luminosity is fully accounted for by their comparatively small mass without appeal to any other physical difference.

4. The current expectation that between density 0·1 and 1 the compressibility of a star will fall off rapidly, as compared with the compressibility of a perfect gas, appears to rest on a false analogy between stellar ions and atoms at ordinary temperature. Owing to the high ionisation, stellar atoms have only about 1/100,000 of the bulk of ordinary atoms, and failure of the laws of a perfect gas is not to be expected till a density 100,000 times higher is reached.

The effect of the high electric charges of the ionised atoms has been considered, but it appears that it would not appreciably affect the compressibility of any of the stars considered.

5. Notwithstanding a wide range of physical condition in the interior of the stars discussed, the ionisation level is not very different in any of them. The assumption that the same molecular weight can be used for all of them is thus closely justified. Attempting a second approximation by taking account of the small variations of molecular weight and of a slowly varying factor in the absorption-coefficient (predicted by Kramers' theory and probable on general grounds), the theoretical curve is scarcely changed for masses greater than $\frac{1}{2}$ and is brought into rather better agreement with observation for the small stars.

6. The extent of the conflict between the present results and the current theory of stellar evolution depends on whether we admit that the mass of a star diminishes to an important extent or not by radiation of energy during its lifetime.

If the mass of the star remains sensibly constant, the statistical diagram of absolute magnitude and spectral type (the "compass-legged" diagram) cannot be interpreted as indicating the course of evolution of a star. Instead, it indicates the locus of equilibrium points reached by stars of different initial mass.

If the star gradually burns itself away in liberating sub-atomic energy, the statistical diagram probably indicates its track of evolution as current theory supposes. In that case the divergence between the present theory and the giant and dwarf theory is narrowed down to the single point, that the diminishing brightness in the dwarf sequence is due to decreasing mass and not to a falling off of compressibility. The conception of an ascending and descending series (judged by effective temperature) is thus retained; although as judged by internal temperature there is probably a continuous ascent.

7. By way of appendix, a discussion is given of the fundamental quartic equation of the theory of radiative equilibrium in which account is taken of the gradual increase of molecular weight from the centre to the boundary of the star.

The Motions of Spiral Nebulæ. By W. M. Smart, D.Sc.

(*Communicated by the Director of Cambridge Observatory.*)

§ 1. Recently Dr. J. H. Jeans has examined (*Monthly Notices*, **84**, 60) the internal motions of spiral nebulæ as determined from the series of measures made by Dr. A. Van Maanen, and has been led from an exploratory discussion of the measures to a tentative, but nevertheless revolutionary, law concerning the forces, which, acting a distance, appear to be necessary to explain satisfactorily the observed movements. Briefly, Jeans finds that the restraining force is proportional, not to r^{-2} as in the Newtonian law, but to $r^{-\frac{1}{2}}$; moreover, the direction in which the force acts depends on the direction of motion of the particular particle on which the force operates.

It is not my purpose in this paper to discuss Jeans' theory, but rather to examine the processes of reduction by means of which are derived the nebular motions, the peculiarities of which the new cosmic force is called in to explain, and, more particularly, to discuss the proper motions of the nebulæ. It is hardly necessary to state that a small systematic or accidental variation in the rather delicately balanced quantities tabulated by Jeans, for example, on p. 62 (*loc. cit.*) would probably necessitate some sort of modification in his results.

§ 2. Van Maanen's measures lead to two important results:—

(*a*) the proper motion of the nebula as a whole;
(*b*) the internal motions.

The former is determined by considering the observed motions of a number of star-like condensations in the nebular arms relative to a group of comparison stars. When the measures of motion of the nebular points have been freed from the proper motion of the nebula, the internal motions are obtained.

In Table I. are collected Van Maanen's results for the proper motions of the 7 nebulæ measured up to the present.

Table I.

Nebula.	Astrophysical Journal.	Number of Comparison Stars.	Number of Nebular Points.	μ_α.	μ_δ.	Probable Errors.	
M. 101	**44**,210	32	87	+0″·005	−0″·013	±0″·002	±0″·002
M. 51	**54**,237	21	79	+0·006	+0·001	±0·003	±0·002
M. 81	**54**,347	14	110	+0·014	−0·005	±0·002	±0·001
N.G.C. 2403	**56**,200	24	76	+0·002	+0·000	±0·001	±0 001
M. 94	**56**,208	14	32	−0·014	+0·000	±0·002	±0·002
M. 63	**57**,49	21	98	+0·005	−0·015	±0·001	±0·001
M. 33	**57**,264	24	400	+0·003	−0·004	±0·001	±0·001

The second column contains the references to Van Maanen's papers in the *Astrophysical Journal*. The third column contains the number of

stars used for the reduction of the nebular motions, and the fourth column contains the number of nebular points for which measures of motion were made. The fifth and sixth columns contain the values of the annual proper motions found by Van Maanen. In the last two columns are certain probable errors to which reference will be made later.

§ 3. For several spirals the proper motion is small, and it is legitimate to inquire what the relation of the probable error is to the small quantities concerned. For the best determined spiral—M. 33—the proper motion is just 0″·5 per century, and a probable error of this amount is not inconceivable on general grounds, that is, without a full discussion of the measures themselves. I do not believe that anyone would be so bold as to question the authenticity of the internal motions —regarded either as rotational or as stream motion—found by Van Maanen; in fact, the more one studies the measures, the greater is the admiration which they evoke. The internal motions are very much larger than the proper motions on the average, and the general symmetry of the former makes them all the more unmistakable. On the other hand, the reality or the comparative accuracy of a small proper motion, derived from a set of measures such as that of M. 33, is hardly evident without further and more detailed discussion, unless we argue that real internal motions imply moderate distance so that the cross velocities of the nebulæ, if of the order of the radial velocities, would necessarily yield proper motions of the order of 0″·5 per century.

An extensive series of measures for the determination of the proper motions of spiral nebulæ was made some years ago by Dr. H. D. Curtis (*Publications of the Astronomical Society of the Pacific*, 27, 214). The 66 spirals which he measured included not only the larger spirals but also several lens-shaped objects—presumably spirals seen edge-on. The average proper motion came out at 0″·033 per annum. It should be noticed that for 10 spirals only the nucleus was measured; in cases where the nucleus was diffuse and difficult to measure, the proper motions were of the order of 0″·06 to 0″·09 per annum. Apparently Curtis was of the opinion that the motions were not real, but that they were due to the accidental errors of the measures. However, Van Maanen (*Astrophysical Journal*, 56, 215) is not of this opinion, but the argument is hardly conclusive. The evidence he brings forward is, briefly, that out of 12 spirals for which proper motions were determined both by Curtis and by himself, the *sign* of the proper motion agreed in 9 cases. But against this must be set the fact that Curtis was unable to detect any sign of rotational motion even in "the better determined nebulæ showing numerous condensations." In the 7 spirals measured by Van Maanen the rotational or stream motion is unmistakable, and is, in magnitude, 2 or 3 times the magnitude of the proper motion—in fact, it is of the order 0″·03 per annum, which is the average value found by Curtis for the proper motion. If Curtis's proper motions were even as much as 50 per cent. in error, it seems hardly likely that the presumably much greater internal motions could have escaped detection. From considerations such as these, I am of the opinion that it is hardly justifiable to lay too much stress on the *magnitude* of the proper motions found by Curtis.

In the light of the greater observational material which Van Maanen has given us, I may anticipate the results of a new analysis of the motions in the 7 spirals by saying here that probable errors have, in general, comparatively little effect on the values of the proper motions given in Table I. and that the reality of the proper motions can hardly be questioned, although the actual values given by Van Maanen may have to be modified to some extent in several cases.

§ 4. Van Maanen's method of reduction is as follows:—From about 20 comparison stars the plate constants are obtained by a least squares solution from the usual equations with quadratic terms, and the proper motions of the n nebular points, relative to the comparison stars, are then derived. These relative proper motions include the rotational or stream components of motions and also the proper motion of the nebula as a whole. When the latter is found, the internal motions—that is, the motions relative to the centre of the nebula—are easily obtained.

From the point of view of the reduction of the measures, it is immaterial whether we consider the internal motions the result of rotational or of stream motion—it is sometimes more convenient to represent the motions in the former way. Let v denote the rotational component and w the radial component of the internal motion of a point on the nebular arm. We shall suppose that the nebula is right-handed. Let θ denote the position angle (measured from the north) of the point and μ_α, μ_δ the components of motion, relative to the comparison stars. If X and Y are the components of proper motion of the nebula as a whole, we have the two equations:—

$$v\cos\theta + w\sin\theta + \mathrm{X} = \mu_\alpha \quad . \quad . \quad . \quad . \quad (1)$$

$$-v\sin\theta + w\cos\theta + \mathrm{Y} = \mu_\delta \quad . \quad . \quad . \quad . \quad (2)$$

Van Maanen adopts three methods of obtaining X and Y.

(*a*) Summing equation (1) for *all* the points, we have, assuming that the points are symmetrically distributed, that is to say, that the polar co-ordinates of the points can be arranged in pairs according to the scheme:—

$$(r_1, \theta_1) \text{ and } (r_1, \theta_1 + \pi)\,;\ (r_2, \theta_2) \text{ and } (r_2, \theta_2 + \pi), \text{ etc.}\,;$$

and assuming further that the motions are symmetrical, *i.e.*

$$v \text{ at } (r_1, \theta_1) = v \text{ at } (r_1, \theta_1 + \pi), \text{ etc.},$$

then

$$n\mathrm{X} = \sum_1^n \mu_\alpha$$

and similarly,

$$n\mathrm{Y} = \sum_1^n \mu_\delta.$$

This method evidently depends on the symmetrical arrangement of *all* the points measured. It does not, however, depend on any assumption as to the variation of v and w with distance from the centre.

(*b*) Dividing the region into the 4 quadrants I.-IV., we have from equation (1)

Mean value of $(v\cos\theta + w\sin\theta)$ in I. + X = mean value of μ_α in I.

For quadrant III. (diammetrically opposite to I.), we have

Mean value of $(v\cos\theta + w\sin\theta)$ in III. + X = mean value of μ_α in III.

If the points in opposite quadrants are symmetrically distributed, and if the motions are again assumed symmetrical, then we have from the above equations

$$2X = \text{mean value of } \mu_\alpha \text{ in I.} + \text{mean value of } \mu_\alpha \text{ in III.}$$

There is a similar equation for II. and IV. and by combining the two pairs of quadrants, we have

$$4X = \text{mean value of } \mu_\alpha \text{ in I.} + \ldots + \text{mean value of } \mu_\alpha \text{ in IV.}$$

Y is obtained from (2) in a similar manner.

(*c*) A symmetrical arrangement is aimed at, and points outside a certain radius are excluded. The derivation of X and Y proceeds as in (*b*).

The values of X and Y so obtained are subtracted from the measured components of proper motion μ_α and μ_δ of the nebular points to give the components of the internal motions, which can then be resolved into rotational or stream motion.

The accuracy of the values of X and Y, as determined from any one of the three methods, depends essentially on the symmetrical distribution of the nebular points. If this symmetry is absent, a systematic error in X and Y results, which is, of course, reflected in the internal motions. It will be seen from the discussion below of the individual nebulæ that there is generally a striking inequality of the number of the nebular points in the 4 quadrants: this adversely affects method (*a*). It does not affect methods (*b*) and (*c*), as here we are dealing with the mean motions in the quadrants, but for (*b*) and (*c*) we have to assume general symmetry in opposite quadrants which is not always possible to achieve unless a *selection* of the points is made.

§ 5. Before proceeding to discuss the question of asymmetry, it is necessary to direct attention for a little to the comparison stars. The proper motion of the nebula is, theoretically, relative to these stars. The magnitudes are given in two cases :—

N.G.C. 2403, $13^{m}\cdot5 - 17^{m}\cdot0$ (p. 201) *

and

M. 94, $15^{m}\cdot5 - 17^{m}\cdot7$ (p. 208).

The mean magnitude of the stars may be presumed to be, for all the spirals, round about 16^{m}. There is very little information at present about the motions of such faint stars. It may be noticed, incidentally, that the Galactic Latitudes of these stars vary from 29° to 77°. A rough estimate from Van Maanen's measures puts the average proper motion

* The page refers to the appropriate volume in Table I.

of the comparison stars at 1″ per century approximately, while the greatest values are of the order of 4″ per century. Three stars of proper motion about 15″ per century had to be excluded from the lists of comparison stars because of this excessive proper motion; two in the region of M. 51 (p. 239) and the third in the region of M. 94 (p. 208). There is thus fairly considerable evidence of motion of these very faint stars, but as to whether it is systematic in any way or no it is difficult, from the measures, to form any idea. It is possible that parallactic motion may account for 0″·1 or 0″·2 per century—or even more—but it would be unsafe to attempt to draw definite conclusions on this point, say by extrapolation in Kapteyn's Tables of Secular Parallax, in which the faintest magnitude concerned is greatly different from the extremely faint magnitudes of the stars with which we are dealing here. Regarding the proper motions of the comparison stars simply as accidental errors, we can estimate the uncertainty, due to this cause, reflected in the values of the components X and Y of the proper motions of the nebulæ. The probable errors are given in the last two columns of Table I. The values are very small, and it is clear that the magnitude of the values of X and Y cannot be seriously disturbed on this ground.

§ 6. I consider now the effect of the asymmetrical distribution of the nebular points on the values of X and Y as derived in method (a). Let the equation of a spiral arm be

$$r = Ae^{\theta \tan \alpha}$$

where α is the angle of the spiral, which we shall suppose right-handed. For the first spiral arm we suppose θ to be measured from the north direction, and for the second arm from the south direction.

If ds is an element of an arm, we have

$$ds = rd\theta \sec \alpha = Ae^{k\theta} \sec \alpha d\theta$$

where

$$k = \tan \alpha.$$

Suppose that we have an ideal distribution of points (N_1 in number) along arm 1 between θ_1 and θ_2, and that in arm 2 we have a like ideal distribution of points (N_2 in number) between θ_1 and θ_3. If θ_3 is greater than θ_2 we have a residue of points in arm 2 between θ_2 and θ_3, which have no corresponding opposite points, and the mean motion (with respect to the total number of points in both arms, that is, N_1+N_2) of this residue will, in consequence, introduce an error into the values of X and Y as derived from method (a).

Let S, T be the components of motion of a point at (r, θ) along the arms and outwards. (These are Van Maanen's stream and transverse velocities.) Then, if S_x, S_y are the components of S in the x and y directions,

$$S_x = \quad S \sin (\theta - \alpha)$$
$$S_y = -S \cos (\theta - \alpha).$$

Similarly,

$$T_x = T \cos (\theta - \alpha)$$
$$T_y = T \sin (\theta - \alpha).$$

Let $\bar{S}_x$, etc., denote the mean values of S_x, etc., from θ_1 to θ_2, or from s_1 to s_2, s_1 being the distance measured along the arm from the centre to the point (r_1, θ_1). Then we have

$$\bar{S}_x = \frac{\int_{s_1}^{s_2} S \sin(\theta - \alpha) ds}{\int_{s_1}^{s_2} ds}.$$

Let $N(\theta_1, \theta_2)$ denote the number of points between θ_1 and θ_2 and $r(\theta_1)$, $r(\theta_2)$, the values of r at θ_1 and θ_2, respectively.

Then,

$$N(\theta_1, \theta_2) = \operatorname{cosec} \alpha [r(\theta_2) - r(\theta_1)]$$

and

$$N(\theta_1, \theta_2).\bar{S}_x = A \sec \alpha \int_{\theta_1}^{\theta_2} S e^{k\theta} \sin(\theta - \alpha) d\theta.$$

If further we assume, for the present, that S and T are constant throughout the spiral arms, we have

$$N(\theta_1, \theta_2).\bar{S}_x = -S[r(\theta_2)\cos\theta_2 - r(\theta_1)\cos\theta_1],$$

similarly,

$$N(\theta_1, \theta_2).\bar{T}_x = T[r(\theta_2)\sin\theta_2 - r(\theta_1)\sin\theta_1].$$

If $\bar{X}(\theta_1, \theta_2)$ is the sum of $\bar{S}_x$ and $\bar{T}_x$, then

$$N(\theta_1, \theta_2).\bar{X}(\theta_1, \theta_2) = r(\theta_2)[-S\cos\theta_2 + T\sin\theta_2] + r(\theta_1)[S\cos\theta_1 - T\sin\theta_1].$$

From the components on the y-axis we derive in a similar manner the equation

$$N(\theta_1, \theta_2).\bar{Y}(\theta_1, \theta_2) = -r(\theta_2)[S\sin\theta_2 + T\cos\theta_2] + r(\theta_1)[S\sin\theta_1 + T\cos\theta_1].$$

For the other arm, on which points are taken as before from θ_1 to θ_3, we have similar equations. Now for both branches we have symmetry between θ_1 and θ_2, but in the second branch there are $N(\theta_2, \theta_3)$ points which remain after the operation described in (α) is completed. The sum of the motions of these points is given by

$$N(\theta_2, \theta_3).\bar{X}(\theta_2, \theta_3) = r(\theta_3)[-S\cos\theta_3 + T\sin\theta_3] + r(\theta_2)[S\cos\theta_2 - T\sin\theta_2],$$

with a similar equation for $\bar{Y}(\theta_2, \theta_3)$.

The total number of points N in both branches is given by

$$N = N(\theta_1, \theta_2) + N(\theta_1, \theta_3).$$

Hence the summation of the equations (1) for all the nebular points gives

$$N.X + N(\theta_2, \theta_3)\bar{X}(\theta_2, \theta_3) = \sum^{N} \mu_\alpha,$$

i.e. X is determined by the method of (a) with an error ϵ_x given by

$$\epsilon_x = \frac{N(\theta_2, \theta_3).\overline{X}(\theta_2, \theta_3)}{N(\theta_1, \theta_2) + N(\theta_1, \theta_3)}.$$

There is, of course, a similar error in Y.

If θ_1, θ_2, $\theta_3 = 0$, π, 2π respectively, we have, numerically,

$$\epsilon_x = \frac{S \sin a[r(2\pi) + r(\pi)]}{r(2\pi) + r(\pi) - 2r(0)}.$$

For $a = 20^\circ$, $S = 0''\cdot025$ per annum,

$$\epsilon_x = 0''\cdot010.$$

For $a = 45^\circ$, $S = 0''\cdot025$,

$$\epsilon_x = 0''\cdot018.$$

This undoubtedly represents an extreme case—one arm is observed uniformly through 360°, whereas the other arm is observed through 180°. In Table II. below I give the results of a few calculations of the values of ϵ_x (the values of ϵ_y will be of a similar nature), taking first, $a = 20^\circ$ (the approximate value for M. 51), and secondly, $a = 40^\circ$ (the value for M. 33), and taking in both cases $S = 0''\cdot025$, $T = -0''\cdot004$, which are, approximately, the average values for M. 33 (p. 274).

TABLE II.

Values of ϵ_x.

	Range of θ.	$a = 20^\circ$.	$a = 40^\circ$.
1.	0 to π		
	0 ,, 2π	0″·010	0″·016
2.	0 ,, 2π		
	π ,, 2π	0·001	0·001
3.	0 ,, 2π		
	$\frac{\pi}{2}$,, 2π	0·001	0·000
4.	0 ,, 2π		
	0 ,, $\frac{3\pi}{2}$	0·006	0·012
5.	0 ,, 4π		
	0 ,, $\frac{7\pi}{2}$	0·006	0 015
6.	0 ,, 2π		
	0 ,, $\frac{7\pi}{4}$	0·004	0·007
7.	0 ,, 2π		
	0 ,, $\frac{9\pi}{4}$	0·001	0·012

The table is by no means exhaustive, but it serves sufficiently well to give an indication of the magnitude of the errors resulting in an indiscriminate application of method (*a*).

So far, S and T have been supposed constant for all distances from the nucleus. Jeans has shown (*M.N.*, **84**, 69) that the dependence of the resultant velocity v on the radius vector r can be pretty well represented by the formula

$$v = Cr^n$$

and from 3 spirals, the value of n is found to be $\frac{1}{4}$. It is on the basis of this value of n that Jeans derives the result that the field force is proportional to $r^{-\frac{1}{2}}$.

Jeans also shows that T can satisfy the formula

$$T = r(a + b \log r).$$

The values of S are approximately the values of the resultant velocity, so that we can write

$$S = Cr^n$$

with a fair approach to accuracy.

As before,

$$N(\theta_1, \theta_2) \cdot \overline{S}_x = C\int_{s_1}^{s_2} r^n \sin(\theta - \alpha) ds$$

$$= \frac{C \sec \alpha}{1 + (n+1)^2 \tan^2 \alpha}\Big[r^{n+1}\{n \tan \alpha \sin(\theta - \alpha) - \cos\theta \sec \alpha\}\Big]_{\theta_1}^{\theta_2}$$

and similarly,

$$N(\theta_1, \theta_2) \cdot \overline{S}_y = \frac{-C \sec \alpha}{1 + (n+1)^2 \tan^2 \alpha}\Big[r^{n+1}\{n \tan \alpha \cos(\theta - \alpha) + \sin\theta \sec \alpha\}\Big]_{\theta_1}^{\theta_2}.$$

From these, that part of ϵ_x which depends on S, can be derived as before. The values will, in general, be higher than in the simpler case ($n = 0$) owing to the index $(n + 1)$ of r. It would be laborious, and perhaps not altogether profitable, to prepare a table similar to Table II. in this more general case and for similar distributions of the nebular points.

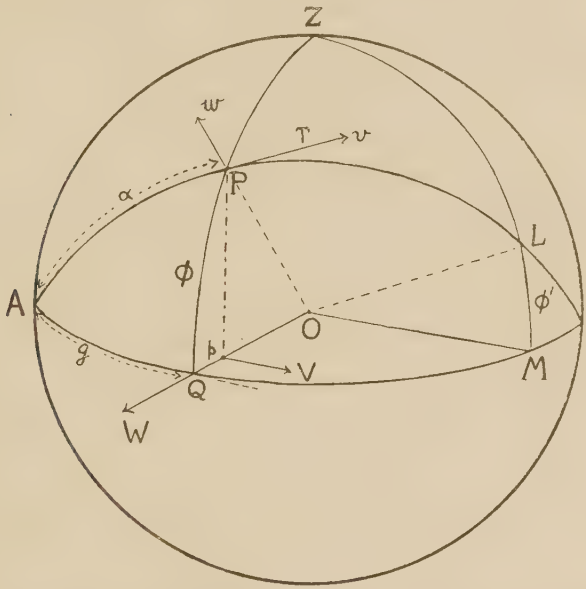
As for T, the general expressions for $N(\theta_1, \theta_2)\overline{T}_x$, etc., are easily obtained, but are rather too complicated for numerical computations. The values of T are much smaller than the values of S, and it would be justifiable to neglect the former altogether in a calculation of this kind.

The main conclusion to be drawn from an indiscriminate use of all the nebular motions as far as method (*a*) is concerned, is that the values of X and Y (the proper motion of the nebula) *may* be subject to considerable errors, sometimes very much greater than the values of X and Y themselves.

It is harder to form an estimate of error when there is asymmetry in opposite quadrants, methods (*b*) and (*c*). The error will hardly be, in general, as considerable as the larger values given in Table II., but,

nevertheless, it is essential to clear the values of X and Y from any suspicion of systematic errors due to the method employed in the reduction of the nebular motions.

§ 7. So far, it has been assumed that the plane of the nebula is perpendicular to the line of sight. The rotational components, v and w in formulæ (1) and (2), for example, can be correlated directly with the actual distances from the centre in this case. In this paragraph are derived formulæ similar to (1) and (2), the inclination of the nebula being taken into account. Let i denote the angle between the plane of the nebula and the tangent plane of the celestial sphere. For M. 33, Van Maanen gives 1 : 1·25 as the approximate ratio of the axes (p. 267), so that i is, approximately, 35°. For M. 81, the value of i is 49°



(p 354). In the latter nebula, the effect of inclination was taken into account in deriving the rotational and stream motions in the plane of the nebula. For N.G.C. 2403, $i = 48°$ (p. 206), and a similar correction was applied. It is interesting to note that for M. 63 the effect of inclination was corrected by means of a specially designed instrument by which a photograph was made of the nebula as a circular object (p. 53).

Let v, w denote as before the rotational components of motion in the plane of the nebula, w being along the radius vector outwards and v at right angles to the radius vector. Let V, W be the observed components in the plane of projection, W being along the projected radius vector and V at right angles.

Suppose a sphere constructed with the centre O of the nebula as centre, and let Z be the pole of the plane of projection. Let P be a point on the nebula. Then w is directed along OP and v along PT. If

the arc PL = 90°, then the direction of v is parallel to the direction OL. The inclination i is the angle PAQ, APL being the plane of the nebula and AQM the plane of projection. A vector directed along OP projects into a vector directed along OQ, and a vector along OL projects into a vector directed along OM.

α, α' denote AP, AL, respectively
g, g' ,, AQ, AM, ,,
ϕ, ϕ' ,, PQ, LM, ,,

It is to be remarked that g and g' are measured from the "major axis" on the tangential plane.

Then the projection of w is $w\cos\phi$ along OQ and the projection of v is $v\cos\phi'$ parallel to OM.

Resolving the latter along and at right angles to OQ and combining the former, we have, as the projections of v and w—

$$(a)\quad w\cos\phi + v\cos\phi'\cos(g'-g) \text{ along OQ.}$$
$$(b)\quad v\cos\phi'\sin(g'-g) \text{ at right angles to OQ.}$$

These, however, are the "rotational" velocities W, V, respectively, observed in the plane of projection, so that

$$\mathrm{V} = v\cos\phi'\sin(g'-g)$$
$$\mathrm{W} = w\cos\phi + v\cos\phi'\cos(g'-g).$$

From these

$$v = \mathrm{V}\sec\phi'\operatorname{cosec}(g'-g)$$
$$w = \mathrm{W}\sec\phi - \mathrm{V}\sec\phi\cot(g'-g).$$

These equations are sufficient to give the values of v and w in the plane of the nebula.

From ALM, noting that $\alpha' = \alpha + 90°$, we have

$$-\sin\alpha = \cos\phi'\cos g'$$
$$\cos\alpha\cos i = \cos\phi'\sin g'$$

so that

$$\mathrm{V} = v\cos\alpha\cos g\cos i(1 + \tan\alpha\tan g\sec i).$$

From APQ, we have

$$\cos\alpha = \cos\phi\cos g$$
$$\tan\alpha = \tan g\sec i.$$

Hence,

$$\mathrm{V} = v\cos i\sec\phi.$$

Again,

$$v\cos\phi'\cos(g'-g) = v(-\cos g\sin\alpha + \sin g\cos\alpha\cos i)$$
$$= -\frac{v}{2}\sin 2g\cos i\tan^2 i\cos\phi.$$

Hence,

$$W = \left(w - \frac{v}{2}\sin 2g \cos i \tan^2 i\right)\cos\phi.$$

From APQ,

$$\tan\phi = \sin g \tan i,$$

so that

$$\sec\phi = +\sqrt{1+\sin^2 g \tan^2 i}.$$

V. Pahlen has given (*A.N.*, **188**, 251) the values of α and $\sec\phi$ corresponding to the values 0°, 10°, . . . 180° of g and the values 15°, 30°, . . . 75° of i.

The essential formulæ are :—

$$V = vC\cos i$$

$$W = \frac{1}{C}(w - vD\sin 2g)$$

$$C = \sec\phi = +\sqrt{1+\sin^2 g \tan^2 i}$$

$$D = \tfrac{1}{2}\cos i \tan^2 i.$$

The measured proper motions of the nebular points, the components X and Y of the proper motion of the nebula, and the velocities V and W are connected by the equations :—

$$V\cos\theta + W\sin\theta + X = \mu_\alpha \quad . \quad . \quad . \quad (3)$$

$$-V\sin\theta + W\cos\theta + Y = \mu_\delta \quad . \quad . \quad . \quad (4)$$

in which θ denotes, as before, the position angle of the nebular point considered. In the formulæ for V and W, g is the angle of the nebular point from the "major axis." Let β be the position angle of that end of the major axis between 0° and 180°. Then

$$g = \theta - \beta.$$

We have, finally, the following equations for v and w :—

$$vC\left[\cos\theta\cos i - \frac{D}{C^2}\sin(2\theta - 2\beta)\sin\theta\right] + \frac{w}{C}\sin\theta + X = \mu_\alpha \quad . \quad (5)$$

$$-vC\left[\sin\theta\cos i + \frac{D}{C^2}\sin(2\theta - 2\beta)\cos\theta\right] + \frac{w}{C}\cos\theta + Y = \mu_\delta \quad . \quad (6)$$

$$C = +\sqrt{1+\tan^2 i \sin^2(\theta - \beta)}.$$

For a symmetrical distribution of n points, the summation of equations (5) and (6) yield, as before, by method (α)

$$nX = \Sigma\mu_\alpha; \quad nY = \Sigma\mu_\delta.$$

When there is lack of symmetry, the same objections apply as noted earlier, and the errors introduced into the values of X and Y would clearly be of the same character and order of magnitude as found in the simpler case.

§ 8. *Methods of Reduction.*—There are two methods of reduction which are not open to the criticism to which I have referred.

(i). A *selection* of nebular points or groups of nebular points is made; the mean motion of a group at (r_1, θ_1) is combined with the mean motion of a similar group at $(r_1, \theta_1 + 180^\circ)$, and so on. If n_1 and n_2 are the numbers of nebular points in each of the pairs of groups, we have from equations (1) and (2) or from equations (5) and (6), two equations of the form

$$2X = \frac{1}{n_1}\sum^{1}\mu_\alpha + \frac{1}{n_2}\sum^{2}\mu_\alpha \quad . \quad . \quad . \quad . \quad (7)$$

$$2Y = \frac{1}{n_1}\sum^{1}\mu_\delta + \frac{1}{n_2}\sum^{2}\mu_\delta \quad . \quad . \quad . \quad . \quad (8)$$

the summations being respectively for groups 1 and 2 of the pair selected. If there are p pairs of groups, we have p equations of the type (7) and (8). It is an advantage in practice to have n_1 and n_2 equal; but this is not always possible unless one or more points of the numerically stronger group are purposely omitted. The process of omission might be suspected to be selective In the applications of this method below, the groups were first selected, whatever n_1 or n_2 might be, and without reference to the values of μ_α and μ_δ; the means were taken and equations (7) or (8) were formed. A weight $(n_1 + n_2)$ was given to each equation. For example, it was found possible to utilise as many as 200 nebular points of M. 33 in this way, divided into 23 pairs of groups. The remaining 200 points, of course, represent an asymmetric distribution in r or in θ. Whether the nebula is inclined or not, the only assumption made in the process is that the values of v and w at (r_1, θ_1) are the same as the values at $(r_1, \theta_1 + 180^\circ)$. An advantage of the process is that we can obtain some idea of the consistency or otherwise of the measures, as represented by the probable errors of X and Y, and some idea as to the validity of the assumption implied in (7) and (8).

(ii). It is not always possible to select an adequate number of nebular points owing to a highly asymmetrical distribution. Van Maanen's results generally show that for points, at a constant distance from the centre in the plane of the nebula, the values of v and w are constant. By dividing the nebula region into a number of rings, circular or elliptical, according as i is zero or not, we can make a least squares solution for v, w, X, and Y for each ring from equations (1) and (2) or from (5) and (6). If i is zero, the labour is not much greater than for (i), but when account has to be taken of the inclination, the amount of calculation is somewhat heavy. It should be noted that v and w are determined both from the measures in x and from the measures in y.

The following paragraphs contain the reductions of the 7 nebulæ —6 by method (i) and 1 (M. 51) by method (ii).

§ 9. *Reduction of M.* 101.—Three Lick plates and 2 Mount Wilson plates were altogether utilised in deriving the proper motion and the

internal motions of the nebula. The results for the proper motion from Van Maanen's reduction are :—

$$X = +0''\cdot005\,; \qquad Y = -0''\cdot013.$$

Van Maanen remarks : "No great weight, however, should be given to this motion of translation, as the hour angles and the parallax factors of the different plates may have vitiated the results" (p. 219).

The distribution of nebular points according to quadrants is :—

N.E. 31 points ; S.E. 16 ; S.W. 22 ; N.W. 18.

Fifty-six points in 15 pairs of groups were used for the reduction by method (i). The results are :—

$$X = +0''\cdot007 \qquad \pm 0''\cdot002$$
$$Y = -0''\cdot014 \qquad \pm 0''\cdot002.$$

The residuals ν_α and ν_δ for the 15 pairs of groups are given in Table III.—Van Maanen's numbers for the nebular points are given in the first column.

TABLE III.

M. 101. *Residuals.*

Groups.	ν_α.	ν_δ.
	″	″
3 ; 63	−0·007	−0·003
4, 5, 36–38 ; 67–69	− 5	− 3
6–9 ; 64–66	+ 6	+ 9
11 ; 53	+ 2	+ 9
12 ; 52	− 22	− 7
14 ; 54	− 20	+ 1
15, 16 ; 55, 57	− 3	+ 17
17, 19 ; 56, 58	+ 17	− 5
20 ; 59, 60	0	− 11
29, 30, 32 ; 71–74	− 11	− 14
31 ; 70	+ 12	− 11
39 ; 82, 83	− 5	− 2
44, 45 ; 85–87	+ 10	+ 5
46 ; 80	− 10	− 4
49, 50 ; 84	+ 17	+ 16

§ 10. *Reduction of M.* 51.—The distribution of nebular points according to quadrants is :—

N.E. 31 points ; S.E. 19 ; S.W. 21 ; N.W. 9.

It was not possible to make a satisfactory application of method (i). It was assumed that $i = 0$ for this nebula. The field was divided up into 3 rings ; ring 1 from $r = 0'\cdot8$ to $r = 1'\cdot6$, ring 2 from $r = 1'\cdot6$ to

$r = 2'{\cdot}2$, and ring 3 from $r = 2'{\cdot}2$ to $r = 3'{\cdot}1$. Two points near the nucleus and 10 points outside ring 3 were thus excluded. Where it was possible, points were grouped so as to reduce the number of equations of condition.

The values of v, w, X, and Y derived from ring 1, with their probable errors, and also the probable error ϵ of a single observational equation, are given in Table IV.

TABLE IV.

Ring 1. *Solutions.*

	Equation (1).	P.E.	Equation (2).	P.E.
v	+0″·012	±0″·002	+0″·020	±0″·002
w	+ 5	2	+ 10	2
X	+ 8	2	...	...
Y	...	...	− 6	1
ϵ	...	7	...	6

The residuals ν_α, ν_δ are shown in Table V. for the points or groups in ring 1. The first column gives the number, as designated by Van Maanen, for each point.

TABLE V.

Ring 1. *Group Residuals.*

Point No.	ν_α.	ν_δ.
16	+0″·011	−0″·007
17	+ 6	− 1
19	− 15	− 5
3, 4, 42	+ 3	+ 5
5, 6, 7	0	− 3
8, 43, 44	+ 1	− 2
45	− 7	+ 10
9	+ 8	− 9
15, 19	− 7	+ 6

The results for ring 2 are as follows:—

TABLE VI.

Ring 2. *Solutions.*

	Equation (1).	P.E.	Equation (2).	P.E.
v	+0″·022	±0″·005	+0″·021	±0″·002
w	+ 11	3	+ 6	2
X	+ 6	3	...	...
Y	...	...	− 1	1
ϵ	...	11	...	5

The residuals are as follows:—

TABLE VII.

Ring 2. *Residuals.*

Point No.	ν_α.	ν_δ.	Point No.	ν_α.	ν_δ.
18	+0″·001	+0″·000	67	+0″·018	−0″·005
60	+ 14	+ 6	46	− 22	− 16
20	− 14	+ 4	47	− 7	+ 10
61	0	− 14	50	− 3	+ 6
62	− 24	− 3	10	+ 30	+ 2
63	0	+ 5	11	− 18	0
66	+ 16	+ 8	13	+ 9	− 3

TABLE VIII.

Ring 3. *Solutions.*

	Equation (1).	P.E.	Equation (2).	P.E.
v	+0″·018	±0″·003	+0″·023	±0″·001
w	+ 11	4	+ 9	1
X	+ 2	2	...	...
Y	...	...	+ 1	1
ϵ	...	13	...	5

TABLE IX.

Ring 3. *Residuals.*

Point No.	ν_α.	ν_δ.	Point No.	ν_α.	ν_δ.
57, 58	+0″·016	+0″·003	23, 64	−0″·003	−0″·011
69, 29	− 5	+ 9	24–26, 65	+ 13	+ 4
30	+ 10	− 3	68	+ 6	− 2
31, 32, 70–72	+ 12	− 3	49	− 17	+ 1
33, 73	+ 3	0	48	− 8	− 6
74	− 25	− 1	28	+ 5	− 3
34, 35	+ 3	− 3	12	+ 22	+ 7
21, 75, 76	− 20	− 4	53, 54	− 14	− 3
36, 37	− 7	+ 11	55, 56	− 18	0
22	+ 3	+ 7			

The results are collected in Table X.

TABLE X.

M. 51. *Collected Results.*

	Ring 1 (16 Points). (1).	Ring 1 (16 Points). (2).	Ring 2 (14 Points). (1).	Ring 2 (14 Points). (2).	Ring 3 (36 Points). (1).	Ring 3 (36 Points). (2).	Weighted Mean.
v	+0″·012	+0″·020	+0″·022	+0″·021	+0″·018	+0″·023	+0″·020
w	+ 5	+ 10	+ 11	+ 6	+ 11	+ 9	+ 9
X	+ 8	...	+ 6	...	+ 2	...	+ 6
Y	...	− 6	...	− 1	...	+ 1	− 2
ϵ	± 7	± 6	± 11	± 5	± 13	± 5	...

The last column contains the weighted means—it is, perhaps, unsatisfactory to assume that v and w are independent of the distance from the nucleus in the light of the results for other nebulæ, but there appears to be little secure evidence from Table X. of any conspicuous variation of v and w with r for M. 51. It is noteworthy that the probable errors from the measures in x are on the average about twice the probable errors in y.

The values of the annual proper motion are :—

$$X = +0''\cdot006 \qquad \pm 0''\cdot002$$

$$Y = -0''\cdot002 \qquad \pm 0''\cdot001$$

as compared with Van Maanen's results of $+0''\cdot006$ and $+0''\cdot001$ respectively.

§ 11. *Reduction of M.* 81.—The nebula is considerably inclined to the line of sight. It was found possible to utilise 71 out of the 110 nebular points to determine the values of X and Y by method (i). The results are :—

$$X = +0''\cdot016 \qquad \pm 0''\cdot002$$

$$Y = -0''\cdot008 \qquad \pm 0''\cdot001$$

as compared with Van Maanen's results of $+0''\cdot014$ and $-0''\cdot005$ respectively. The group residuals v_a, v_δ are given in Table XI.

TABLE XI.

M. 81. *Residuals.*

Group.	ν_α.	ν_δ.
	″	″
32, 81 ; 96	+0·016	+0·002
26, 63 ; 92	+ 5	+ 2
67 ; 95	+ 3	+ 2
37, 38, 77, 78 ; 6–8, 97	+ 10	0
9–15, 19 ; 39, 72, 74–76	+ 2	− 1
71 ; 108, 109	− 2	+ 7
17, 18, 20, 21, 102 ; 41, 43–45	− 4	+ 1
64 ; 93	− 12	0
29, 66 ; 94	+ 1	− 24
46, 86 ; 107	− 18	+ 10
5, 104, 105 ; 83–85, 110	− 3	− 2
24 ; 55, 91	− 18	0
22 ; 52, 53, 89, 90	− 3	+ 4
73 ; 98, 101	+ 11	+ 5
47 ; 106	+ 8	+ 7
82 ; 100	+ 5	− 13

The accuracy and consistency of the measures are both alike remarkable. The distribution of nebular points according to quadrants is:—

N.E. 23 points; S.E. 35; S.W. 10; N.W. 36.

§ 12. *Reduction of N.G.C.* 2403.—The 76 nebular points are distributed in the quadrants thus:—

N.E. 10 points; S.E. 24; S.W. 15; N.W. 27.

Forty-four selected points, divided into 11 pairs of groups, gave the following solution by method (i):—

$$X = +0''{\cdot}001 \qquad \pm 0''{\cdot}001$$
$$Y = +0''{\cdot}001 \qquad \pm 0''{\cdot}001$$

as compared with Van Maanen's results $+0''{\cdot}002$ and $0''{\cdot}000$ respectively. The proper motion of the nebula is, in either case, almost negligibly small. The residuals for the 11 pairs of groups are given in Table XII.

TABLE XII.

N.G.C. 2403. *Residuals.*

Group.	ν_α.	ν_δ.
	″	″
3–7 ; 68–70	−0·006	+0·001
9, 10 ; 65, 66	+ 1	− 8
12–14 ; 61–64	− 2	+ 1
15, 27 ; 54	− 1	− 5
16 ; 67	0	− 12
17–19 ; 51–53	+ 7	+ 6
20, 21 ; 46–48	+ 3	+ 4
22 ; 45	+ 4	− 10
28, 30 ; 43	+ 1	+ 2
32 ; 55	− 4	+ 5
39 ; 57	− 4	− 1

§ 13. *Reduction of M.* 94.—The number of nebular points for which measures were made is much smaller than usual. Van Maanen remarks (p. 210) that "the measures were, as a whole, rather difficult." The distribution of the 32 points in the quadrants is:—

N.E. 8 points; S.E. 10; S.W. 8; N.W. 6.

It was found possible to utilise as many as 26 points in the reduction by method (i). The results for X and Y are:—

$$X = -0''\cdot015 \qquad \pm 0''\cdot002$$
$$Y = +0''\cdot003 \qquad \pm 0''\cdot002.$$

The results given by Van Maanen are $-0''\cdot015$ and $0''\cdot000$ respectively. The residuals for the 7 pairs of groups are as follows:—

TABLE XIII.

M. 94. *Residuals.*

Group.	ν_α.	ν_δ.
	″	″
1–4 ; 15–19	+0·002	−0·002
5 ; 22	− 15	+ 15
6 ; 31	+ 4	0
8, 9 ; 28, 29	− 4	+ 5
10 ; 23, 24	+ 9	0
11, 12 ; 25, 26	+ 4	− 9
14 ; 27	− 8	+ 6

§ 14. *Reduction of M.* 63.—The 98 nebular points are distributed in the quadrants as follows:—

N.E. 19 points; S.E. 33; S.W. 20; N.W. 22.

Sixty-five points divided into 15 symmetrical pairs of groups were analysed by method (i). The derived values of X and Y are:—

$$X = +0''\cdot007 \qquad \pm 0''\cdot0006$$
$$Y = -0''\cdot015 \qquad \pm 0''\cdot0004.$$

Van Maanen's results are $+0''\cdot005$ and $-0''\cdot015$ respectively. It will be noticed that the probable errors are remarkably small. The residuals for the 15 pairs of groups are shown below:—

TABLE XIV.

M. 63. *Residuals.*

Group.	ν_α.	ν_δ.
1–3; 87, 88, 92	$-0''\cdot002$	$+0''\cdot001$
4, 5; 79, 90	− 1	+ 3
7; 74, 85	− 2	+ 2
8, 13–15; 75, 78, 80	0	0
9–12; 73, 76, 77, 84	− 1	− 2
16, 17; 70	− 5	+ 4
18, 19; 68, 69	+ 6	0
22; 64	+ 7	+ 1
23, 39; 56	− 1	− 2
24, 33; 53, 65	+ 6	0
25, 32; 61	− 1	+ 4
26, 27; 52, 66	− 7	− 2
28–30; 62, 63	+ 3	− 5
34, 35; 54, 55, 57	+ 1	+ 1
38, 40; 41, 42	− 1	+ 1

§ 15. *Reduction of M.* 33.—The distribution of the 400 nebular points according to quadrants is:—

N.E. 97 points; S.E. 88; S.W. 140; N.W. 75.

These counts include the nebular point, No. 367, which showed a large proper motion and was not included further in the reductions (p. 267). 200 points, arranged in 23 pairs of symmetrical groups, were reduced by method (i). Van Maanen does not give the measured values of μ_α, μ_δ for the points; however, he gives the values of μ_α, μ_δ relative to the centre of the nebula, his values of X and Y having been applied in the usual way. The analysis of the 200 points will consequently yield the corrections (if any) to his values $+0''\cdot003$, $-0''\cdot004$ of X and Y.

The values of the components of proper motion which I have derived are:—

$$X = +0''\cdot0034 \qquad \pm 0''\cdot0005$$
$$Y = -0''\cdot0044 \qquad \pm 0''\cdot0004.$$

The results are identical with those of Van Maanen—the insignificance of the probable errors is remarkable, and is eloquent testimony to the extraordinary precision of the measures.

The group residuals are shown in Table XV.

TABLE XV.

M. 33. *Residuals.*

Groups.	ν_α.	ν_δ.
	″	″
23–26 ; 390–393, 399	−0·002	−0·004
39–41 ; 315–320	+ 4	− 1
47–57 ; 366, 368, 375–378	+ 8	+ 2
58–61 ; 359–362	+ 1	− 3
76, 79–82 ; 356, 357	− 5	− 1
338–345 ; 83, 88–94, 166–168	+ 2	0
95–97 ; 369, 370	+ 5	− 2
98, 99 ; 331, 332	+ 7	+ 3
100–102, 149, 150 ; 327, 329	+ 1	− 1
103, 104 ; 321, 322	0	− 2
110–116 ; 306–314	− 2	− 2
117–124 ; 296, 297, 302–305	− 2	− 5
147, 148, 153 ; 274–276	+ 2	+ 4
156–158 ; 271–273	+ 1	+ 5
159–165 ; 268, 330, 333–337	+ 5	+ 2
177–183 ; 250–255	− 6	− 2
190–197 ; 224–227, 234–238	− 4	− 2
198 ; 258, 260	0	+ 1
199–201 ; 263–265	0	− 2
202 ; 267	+ 6	+ 2
203 ; 270	+ 2	+ 1
206–208 ; 277, 280, 281	− 1	+ 3
214–219 ; 283–287	+ 2	+ 3

§ 16. The results of the preceding reductions are in general agreement with Van Maanen's results. The nature of the probable errors leaves very little doubt that the proper motions are real and of the right order of magnitude. The mean proper motion of the 7 spirals is approximately 0″·01 per annum, and on the assumption that the mean cross velocity is the same as the mean radial velocity of the 30 or 40 nebulæ for which the radial velocities are known (600 kms. per second), the mean parallax of the 7 spirals is approximately 0″·0001.

§ 17. *Proper Motions of the Nebulæ in Galactic Co-ordinates.*—The well-established fact that the radial velocities of the spiral nebulæ show that these objects are in general receding from the Galaxy (the exceptions

are noted by Mr. J. H. Reynolds, *Observatory*, 45, 268) ought to find, one would expect, support from the proper motions. The material with which we are dealing here is, of course, rather meagre; in particular, the proper motion of N.G.C. 2403 is so small that very little reliance can be placed on what the figures show for this nebula; also, the results for M. 94 are based on the measures of a comparatively small number of nebular points, which apparently are not so well defined as in other nebulæ, and perhaps one would not unduly press the results for this nebula.

In Table XVI. I give the components μ_λ, μ_β of the proper motion, parallel to the galactic plane and towards the north pole of the Galaxy respectively, derived from Van Maanen's results and also from those found above. The galactic longitude and latitude (λ and β) are given in the second and third columns.

TABLE XVI.

Proper Motions in Galactic Co-ordinates.

	λ.	β.	Van Maanen μ_λ.	Van Maanen μ_β.	μ_λ.	μ_β.
	°	°	″	″	″	″
M. 101	67	+59	−0·014	+0·002	−0·016	+0·001
M. 51	67	+68	− 4	− 4	− 6	− 1
M. 81	108	+41	− 4	+ 14	− 4	+ 17
N.G.C. 2403	118	+29	0	+ 2	− 1	+ 1
M. 94	86	+77	+ 14	+ 1	+ 15	− 3
M. 63	67	+73	− 7	+ 14	− 9	+ 14
M. 33	102	−31	+ 4	− 3	− 4	− 3

The signs of μ_β in the fifth column (Van Maanen's proper motions) show that, with the exception of M. 51, the nebulæ are moving *away* from the galactic plane (the negative sign in the case of M. 33, which is in the southern galactic hemisphere, means, of course, motion away from the Galaxy). In my results—seventh column—M. 94 is an additional exception, but, on account of the reasons stated above, it may not be advisable to attach too much importance to this exception. The best determined nebulæ are undoubtedly M. 33, 63, 81, and for these the velocities away from the Galaxy are unmistakable.

The radial velocity of M. 81, as stated by Reynolds (*loc. cit.*), is − 30 kms. per second—it is one of the exceptions referred to above. Its cross velocity is almost at right angles to the galactic plane, its galactic latitude is moderate, and its proper motion is the second largest of the 7. If its parallax is of the order 0″·0001, these facts would seem to show that the space-velocity of this nebula is also away from the galactic plane and that it is not an exception to the almost general phenomenon of recession.

Observatory, Cambridge.

An Astrophysical Determination of the Average Life of an Excited Calcium Atom. By E. A. Milne.

(*Communicated by the Director of the Solar Physics Observatory, Cambridge.*)

1. It is generally accepted that the atoms in the high-level regions of the sun's chromosphere are supported by radiation pressure. In this note it is shown that the hypothesis of a steady state of equilibrium under radiation pressure combined with Einstein's theory of radiation leads to a precise evaluation of the average life of an excited atom.

2. At the total eclipse of 1905, radiation corresponding to the H and K lines of calcium was observed by Mitchell * up to a height of 14,000 kms. above the limb of the sun. The next highest elevation was given by some of the Balmer lines, which extended up to 8500 kms., and the g-line of calcium extended only to 6000 km. We infer from this that the gas above 8000 kms. consists largely, if not entirely, of calcium atoms; and further, following Saha, that the calcium atoms are all ionised. A simple calculation † shows that the pressure must be extremely low, perhaps as low as 10^{-13} atmosphere. In spite of this low pressure, there will be scarcely any appreciable amount of second-stage ionisation. In the first place, even in thermodynamic equilibrium, at this pressure and at a temperature of 5000°, second-stage ionisation would be only just beginning. But in the second place, the state in these high levels must be far removed from one of thermodynamic equilibrium, and the usual formula for calculating the ionisation is invalid; the state of ionisation must be determined by a specific calculation of the number of ionisations per second and the rate of recombination. Now, collisions are so rare that ionisation by collision is out of the question; photo-electric ionisations are practically impossible because the sun is too deficient in ultra violet radiation of frequency high enough to remove the second electron; and the effect of cumulative ionisation by radiation can be neglected. The high-level chromosphere may therefore be taken to consist of singly ionised calcium atoms and an equal number of free electrons.

The atoms are emitting radiation of wave-lengths λλ3933, 3968. These lines are the first pair of the principal series of Ca^+, and their series specification is $(1\sigma - 1\pi_{1,2})$. They originate from transitions from the $1\pi_1$ and $1\pi_2$ states (4_2 orbits) to the normal 1σ-state (4_1 orbits). No other Ca^+ radiation has been recorded in the high-level chromosphere, nor would any be anticipated. We may therefore suppose that the $1\pi_{1,2}$ and 1σ states are the only ones that come into consideration; a Ca^+ atom in the high-level chromosphere is either in the 1σ-state or in one of the 1π-states.

It cannot be in both the 1π-states simultaneously, nor can it undergo a transition from one of them to the other. We shall therefore introduce a simplification by regarding the ionised atoms as capable of only a single 1π-state, and we shall choose the energy-value of this state so that the difference $(1\sigma - 1\pi)$ corresponds to the mean wave-length λ3950.

* *Astrophys. Journ.*, 38, 407, 1913. † *M.N.*, 83, 418, 1923.

Consider first the mechanical equilibrium. The mass of the free electrons is small, and provided we ensure that the ionised calcium atoms are supported against gravity, the electrostatic forces of attraction will be sufficient to support the free electrons against gravity.

Each time a Ca^+ atom absorbs a quantum, it undergoes a transition from the 1σ-state to the 1π-state, and at the same time acquires momentum of amount $h\nu/c$, h being Planck's constant, c the velocity of light, and ν the frequency corresponding to $\lambda 3950$. So long as it remains in the 1π-state it has no opportunity of further absorption, and thus no opportunity of responding further to the pressure of radiation; it will therefore fall freely towards the sun, under gravity. After some interval it will undergo the return transition to the 1σ-state, emitting a quantum of ν-radiation and acquiring the corresponding momentum of recoil. It is now ready to absorb a second quantum of ν-radiation, and the process will be repeated.

The atom will be exposed to radiation only on the side towards the sun—we are considering heights so great that the back radiation from the few atoms which are still higher will be negligible. Consequently the momentum acquired in absorptions will be principally outward momentum. On the other hand, emissions will be in a random direction, and consequently the recoil momentum acquired in emissions will on the average be zero.

Let τ be the average time the atom spends in the 1π-state—the excited state; and let τ' be the average time the atom spends in the 1σ-state—the normal state. The time occupied by the act of transition, in either direction, is assumed to be negligible * in comparison with τ or τ'. Then on the average the complete cycle consisting of one absorption and one emission will occupy a total time of $\tau+\tau'$, and during this time the momentum acquired from the pressure of radiation is that due to the absorption of one quantum. For equilibrium this must be equal to the momentum acquired in the same time $\tau+\tau'$ in falling freely under gravity. Equating the two we have an equation for $\tau+\tau'$.

Consider next the interchange of radiation. The time τ, according to current theory, is a natural constant of the atom, or rather of the state 1π. The time τ', on the other hand, is inversely proportional to the intensity of ν-radiation to which the atom is subject; if the intensity is doubled, the chance of absorbing a quantum is doubled and τ' is halved. The times τ and τ' are connected with the probabilities of emission and absorption, and Einstein † has shown how the ratio of these probabilities can be obtained from a study of thermodynamic equilibrium. His result is, in effect, Kirchhoff's law in quantum form. Einstein's formula for the ratio of the probabilities can be applied to obtain τ/τ' for a system of atoms which is not in thermodynamic equilibrium. Applying it to the ionised calcium atoms in the high-level chromosphere, we evaluate

* Einstein's theory of radiation has been criticised by Bohr, partly on the ground that it makes this assumption. The assumption seems, however, to be in accord with general theory, and, moreover, an assumption of this kind is fundamental in the treatment of any problem by statistical mechanics, namely, the assumption that the times occupied by the constituent systems in exchanging energy are small compared with the intervals during which they retain their energy unchanged.

† *Phys. Zeit.*, 18, 121, 1917.

τ/τ' in terms of the intensity of ν-radiation to which the atoms are subject. This is, of course, the residual intensity in the centre of the observed H and K absorption lines.

Combining the equation for $\tau+\tau'$ with the equation for τ/τ' we obtain τ and τ' separately. The value of τ' is of no particular interest, for it depends on the chromospheric radiation-intensity. But τ is an atomic quantity, and if the method is sound the resulting value should be independent of the various solar quantities used in deducing it.

3. We now give the mathematical details. Let $A_{2\to1}$ be the probability that an atom in a given excited state 2 will during the interval dt emit a quantum of radiation and return to the state 1. Let $B_{1\to2}I_\nu d\nu dt$ be the probability that an atom in the state 1, exposed to *isotropic* ν-radiation of intensity $I_\nu d\nu$, will in time dt absorb a quantum and change to the state 2. Then Einstein's relation is *

$$A_{2\to1}=\frac{2h\nu^3 d\nu}{c^2}B_{1\to2} \quad . \quad . \quad . \quad . \quad (1)$$

The probability coefficients are atomic quantities, strictly independent of statistical variables such as the temperature, and thus relation (1), though deduced by consideration of thermodynamic equilibrium, will be true always.

From the definitions, the mean life τ in the excited state is given by †

$$\tau=\frac{1}{A_{2\to1}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Exposed to isotropic radiation, the mean life in the normal state would be given by

$$\tau'=\frac{1}{B_{1\to2}I_\nu d\nu}.$$

In the high-level chromosphere, however, the atoms are exposed to radiation on one side only. Let us assume that this is uniformly distributed over the hemisphere facing the sun. Then τ' for an atom in the high-level chromosphere is given by

$$\tau'=\frac{1}{\frac{1}{2}B_{1\to2}I_\nu d\nu} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining (2) and (3) and using (1) we have

$$\frac{\tau}{\tau'}=\tfrac{1}{2}I_\nu\frac{c^2}{2h\nu^3} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now suppose the intensity in the centre of the lines is some fraction r of the intensity in the adjacent continuous spectrum. Assuming the

* Einstein, *loc. cit.* For the introduction of the frequency interval $d\nu$, see Milne, *Phil. Mag.*, 47, 216, 1924, where a proof of formula (1) in the present notation is given *de novo*. The statistical weights of the states 1 and 2 have been taken as equal.

† The stimulated emission is here neglected. Its ratio to the unstimulated emission is $\frac{1}{2}r(e^{h\nu/kT}-1)^{-1}:1$, which is in all cases very small.

photospheric radiation is that of a black body * at temperature T, we have the formula

$$I_\nu = r\frac{2h\nu^3}{c^2}\cdot\frac{1}{e^{h\nu/kT}-1} \quad . \quad . \quad . \quad . \quad (5)$$

Our assumption of radiation distributed equally over the hemisphere is equivalent to neglect of the change of intensity between centre and limb, and we therefore adopt for T the effective temperature as deduced from the integrated radiation from the whole disc. We shall take $T = 6000°$. Combining (4) and (5) we have

$$\frac{\tau}{\tau'} = \frac{\frac{1}{2}r}{e^{h\nu/kT}-1} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $0 < r < 1$.

When a quantum is absorbed in a direction making θ with the vertical, the vertical component of momentum is $(h\nu\cos\theta)/c$. The average value of $\cos\theta$ over the hemisphere is $\frac{1}{2}$, hence the average momentum acquired for each quantum absorbed is $\frac{1}{2}h\nu/c$. During the time $\tau+\tau'$ the momentum acquired under gravity is $mg(\tau+\tau')$, where m is the mass of the calcium atom. Equating these, we have

$$mg(\tau+\tau') = \tfrac{1}{2}h\nu/c \quad . \quad . \quad . \quad . \quad (7)$$

Equations (6) and (7) are the equations required.

For ν corresponding to $\lambda 3950$, we find $h\nu/kT = 6{\cdot}05$. Inserting the numerical values $m = 40 \times 1{\cdot}65 \times 10^{-24}$ gram, $g = 2{\cdot}73 \times 10^4$ cm. sec.$^{-2}$, we find from (6) and (7)

$$\tau/\tau' = r \times 1{\cdot}18 \times 10^{-3} \quad . \quad . \quad . \quad . \quad (8)$$

$$\tau+\tau' = 4{\cdot}6 \times 10^{-5} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (8) shows that τ is small compared with τ'. Neglecting it in (9), and then using (8), we find

$$\tau = r \times 5{\cdot}4 \times 10^{-8} \text{ sec.}$$

Since $r < 1$, we have immediately

$$\tau < 5{\cdot}4 \times 10^{-8} \text{ sec.} \quad . \quad . \quad . \quad . \quad (10)$$

The time τ is difficult to measure experimentally, but direct measurements using canal rays have yielded values of the order of 10^{-8} sec. for various lines of various elements.† Thus Wien found $\tau = 2 \times 10^{-8}$ for the hydrogen lines Hβ and Hγ. No measures have been made for ionised calcium, but result (10) is seen to be in satisfactory agreement with the general experimental results.

On the present theory, an exact measurement of the ratio r of the intensity at the centre of the H and K lines to the intensity in the adjacent continuous spectrum would furnish an exact determination

* The recent observations of Fabry and Buisson (*C.R.*, 175, 156, 1922), and of H. H. Plaskett (*Pub. Domin. Astrophys. Obs.*, 2, 253, 1923), show that between the absorption lines the sun radiates very closely like a black body.

† For a summary of the work, see Foote and Mohler, *The Origin of Spectra*, p. 96.

of τ. The matter is rendered difficult by the known complexity of structure often found in the centres of the H and K lines—the bright H_2 and K_2 reversals and the dark central H_3 and K_3—of which the present theory takes no account. Measures of the intensity distribution in the neighbourhood of the H and K lines at the centre and limb of the sun have, however, been made by Schwarzschild.* The dispersion, 1 Å = 0·4 mm., was sufficient to give a fair idea of the intensity distribution, though insufficient to show the central detail. The measures were made by estimating the photographic blackening, and the results are given in the form of curves with wave-lengths as abscissæ and intensities expressed in magnitudes as ordinates. From the curves it appears that at the centre the intensity-ratio is about 2·6 magnitudes, at the limb about 2·05 magnitudes. If we adopt 2·4 magnitudes as a mean value for the whole disc, this gives $r = 0{\cdot}11$, whence

$$\tau = 0{\cdot}6 \times 10^{-8} \text{ sec.} \quad . \quad . \quad . \quad . \quad (11)$$

The value of τ for an excited atom of ionised calcium is thus determined from astrophysical data.

4. It will be noticed that no knowledge of the *breadths* of the lines has been required. The method proceeds by elimination of the troublesome frequency-interval $d\nu$. This shows that no correction is required to allow for the fact that two lines are concerned—the treatment amounts to that for a single line of double the breadth, and the actual breadth is immaterial. The breadth $d\nu$ is of course of importance in attempting to calculate the total mass that can be supported by radiation pressure, or to calculate the absorption coefficient. The mass absorption coefficient k_ν is given by †

$$k_\nu = \frac{B_{1\to 2} h\nu}{4\pi m} = \frac{e^2}{8\pi m \tau \nu^2 d\nu},$$

which requires $B_{1\to 2}$ to be separated from $d\nu$ or alternatively involves an estimation of $d\nu$. The present method of using τ and τ' thus avoids the uncertainties that arose in a previous attempt ‡ to deal with the equilibrium of the high-level calcium vapour. For $d\nu$ corresponding to $d\lambda = 0{\cdot}1$ A, the value found in this paper for τ gives $k_\nu = 8 \times 10^9$ for the absorption of H and K radiation by Ca^+.

It may be added that though we have presented the argument in §§ 2, 3 as referring to the equilibrium of a single atom, each atom will in fact undergo considerable fluctuations of motion, corresponding to fluctuations in the rate of acquisition of momentum by absorption of radiation. But a rigorous statistical discussion leads to equations which reduce essentially to (6) and (7).

If it be urged against our result that there is no positive reason for believing the sun's chromosphere to be in a steady state, the plausibility of the value we have deduced for τ shows that there is at least nothing impossible or inconsistent in the assumption.

It has been assumed that radiation pressure and gravity are the only

* *Berlin Sitz.*, 1914, p. 1198.

† *Phil. Mag.*, **47**, 217, 1924.

‡ *M.N.*, **83**, 418, 1923.

forces acting on the atoms. If the sun as a whole possessed an electrostatic charge, a correction would have to be applied to the value of g used. Escape of electrons from the chromosphere into space, due to velocities of thermal agitation, will endow the sun with a positive charge, and, gains or losses in other ways being ignored, this will go on until the further loss of electrons is checked by the electrostatic attraction of the acquired positive charge. It has been shown elsewhere * that on simple gravitational equilibrium the maximum positive potential that can be attained by the sun by the escape of electrons is 30 volts. The repulsion exerted by this on a positively charged calcium atom amounts to only one part in 2500 of the attraction due to gravity. Under radiation pressure, however, the escape of atoms—always negligible in gravitational equilibrium—will be facilitated, for an outward moving atom will continue to experience the pressure of radiation. Consequently the positive charge arising from the escape of electrons will be partly neutralised by the escape of positively charged atoms, and the residual charge will be even smaller than that calculated above. The correction is thus negligible. Presumably a steady state will be set up, in which electrons and ions escape comparatively freely in equal numbers, the extra mobility of the electrons being compensated by the field of the residual positive charge.

It must be admitted that the state of affairs in the boundary region is somewhat obscure, for if radiation pressure exactly balanced gravity, every outward moving atom would be able to escape. The standard investigations of the escape of molecules from atmospheres in gravitational equilibrium require modification before any detailed calculation can be made of the actual escape from an atmosphere supported by radiation pressure. But difficulties concerning the escape of molecules, and other questions connected with mean free path phenomena, do not invalidate the investigation of equilibrium given in the present paper; the escape of molecules depends on velocities, the equilibrium depends on accelerations, and the present investigation obtains the condition for equilibrium in the limit when gravity and radiation pressure balance exactly at great heights. The extent of the solar chromosphere suggests that here the limiting condition must be very closely approached.

5. From (6) and (7) we obtain a formula for τ,

$$\tau = \frac{r}{4mgc} \cdot \frac{h\nu}{e^{h\nu/kT} - 1} \quad . \quad . \quad . \quad . \quad (12)$$

It follows that for a given atom with a given resonance frequency ν and given value of τ, chromospheric equilibrium of the kind here considered is only possible if the ratio of the intensity in the centre of the absorption line to the intensity in the neighbouring continuous spectrum has the value r_0 given by

$$r_0 = \frac{4mgc\tau}{h\nu}(e^{h\nu/kT} - 1) \quad . \quad . \quad . \quad . \quad (13)$$

If the right-hand member of this equation is greater than unity, high-level chromospheric equilibrium is not possible simultaneously

* *Camb. Phil. Trans.*, 22, 514, 1923.

with the presence of an absorption line;* this situation will occur if the atom is too heavy, if g is too large, if ν/T is too large, or if τ is too low. For Ca^+ with the value of τ found, the critical temperature below which high-level chromospheric equilibrium is not possible is found to be 4400° for a star with the same value of g as the sun.

If the right-hand side of (13) is less than unity, chromospheric equilibrium may or may not be possible. The gas in the reversing layer will have given rise, in general, to the formation of an absorption line, say with intensity-ratio r_1. If r_1 is less than the critical value r_0 given by (13), chromospheric equilibrium will not be possible. If $r_1 = r_0$, chromospheric equilibrium is just possible, although the chromosphere will contain an infinitesimally small number of atoms. If $r_1 > r_0$, chromospheric equilibrium will be possible, and the chromosphere will have such a thickness as will reduce the reversing layer intensity r_1, by absorption, to r_0. In this third case it is easy to picture the mechanism of formation of the chromosphere. If we imagine for a moment the chromosphere removed, radiation pressure at the confines of the reversing layer will exceed gravity, and many of the atoms will be swept outwards. As more and more are driven off, those driven off later will act as a shield between the atoms earlier driven off and the radiation from the reversing layer, and the process will go on until a chromosphere has been brought into existence of a thickness just sufficient to reduce the residual intensity to r_0. No steady state is possible so long as the residual intensity in the upper levels exceeds r_0.†

For the sun, our adopted value $r_0 = 0{\cdot}11$ gave $\tau = 0{\cdot}6 \times 10^{-8}$ sec. for Ca^+. Using this value of τ, we can calculate r_0 for other stars for the H and K lines. The following short table gives the values of r_0 for Capella, Sirius, and J. S. Plaskett's massive O-star. The values of g have been calculated from the masses and densities, using Eddington's ‡ figures for the first two and Plaskett's § for the third. Two alternative temperatures are given for the O-star.

Star.	M(⊙).	ρ.	g.	T.	r_0.
Capella	4·2	0·0026	$0{\cdot}067 \times 10^4$	5,500°	0·010
Sirius	2·43	0·455	$1{\cdot}73 \times 10^4$	11,000	0·0043
O-Star	86	0·01	$0{\cdot}56 \times 10^4$	18,000	0·00035
				24,000	0·00019

From the table it is apparent that for early-type stars, the residual intensity must be reduced to a very small fraction of the photospheric

* For a bright line, $r > 1$, and chromospheric equilibrium may then still be possible. A bright line in the integrated spectrum of a star may, however, itself be due to an extensive "chromosphere" (as Huggins pointed out), perhaps comparable in thickness with the star's radius. But the appropriate value for r is then not the observed intensity-ratio but the ratio corrected for the different radiating areas of star and chromosphere, and even if the observed ratio is greater than unity the corrected value of r may be less than unity. However, the formation of bright lines is so little understood that we omit any further consideration here.

† It is easily shown that the actual value of the residual intensity in the line (not the intensity-ratio), for equilibrium under radiation pressure, is $8\nu^2\tau\, mg/c$, which is independent of T.

‡ *M.N.*, **84**, 109, 1924; *M.N.*, **83**, 105, 1922. § *M.N.*, **82**, 455, 1922.

intensity before a steady state is possible. It may be that for such stars the screen of calcium atoms never becomes sufficiently thick to reduce the intensity-ratio to these values. In that case, atoms of ionised calcium will be continually expelled under radiation pressure. The possibility of this has been previously suggested by many writers, but the present calculations embody the idea in definite quantitative form.

We have so far considered simply the equilibrium of the chromosphere. Clearly, to give a definite answer to the question of the expulsion of calcium atoms under radiation pressure, an exhaustive discussion of the *stability* of the equilibrium would be required. The following simple consideration suggests that high-level chromospheric equilibrium may be slightly unstable. Consider a single calcium atom some way outside a chromosphere in equilibrium. We may regard it as a small sphere of a certain radius, whose cross-section is equal to the "effective cross-section for absorption."* The amount of radiation incident on it and absorbed will fall off as the inverse square of its distance from the centre of the star, and the value of g will fall off according to the same law. To a first approximation, therefore, the equilibrium is neutral. But at the greater distances the radiation is confined in a smaller solid angle, and hence absorption will result in the communication of a proportionally larger component of outward momentum—the average value of $\cos\theta$ will be larger. Thus, if the atoms in the high-level chromosphere are just in equilibrium, for an atom considerably outside the chromosphere the pressure of radiation would slightly exceed gravity. The equilibrium would therefore be unstable.

6. The simple procedure of the present paper applies solely to the first principal line (or lines) of a neutral or ionised atom whose next-stage ionisation-potential is so high that ionisation or further ionisation respectively does not occur. The first principal pairs of Ca^+, Sr^+, and Ba^+ seem to be the only lines that satisfy this criterion and at the same time lie in the region of appreciable intensity. A few remarks on other atoms may be of interest.†

The resonance lines of Na, the D-lines, are found in the chromosphere only up to a height of 1000 kms. As Saha first showed, the ionisation potential (5·1 volts) is so low that above this height no appreciable proportion of Na atoms would remain un-ionised. But the chief lines of Na^+ lie in the far ultra-violet, in a region where the solar radiation is too weak to exert any appreciable pressure. It appears probable, therefore, that Na atoms which chance to become ionised will not be supported by radiation pressure; except in so far as they are supported against gravity by an appreciable pressure gradient, they must fall back towards the sun until they recapture an electron with which to absorb. A height of 1000 kms. is probably the limiting height of the invisible Na^+ atoms as well as of the visible Na atoms.

Of a different interest are the atoms of hydrogen and helium. Both have high ionisation potentials, but the Balmer lines and the ordinary arc lines of helium are not derived from the normal states of the respective

* *E.g.* Kramers, *Phil. Mag.*, **46**, 836, 1923.

† These paragraphs develop ideas originally put forward by Saha, *Nature*, **107**, 488, 1921.

atoms. The resonance lines are far in the ultra-violet. A hydrogen atom which has just emitted a Balmer line may for its next transition either absorb a Balmer line or emit the first Lyman line, $\lambda 1216$. If it does the latter, it will presumably have to wait a considerable time before its next absorption, owing to the poverty of the solar radiation in the requisite region of frequencies. Statistically, therefore, the momentum acquired in virtue of Balmer absorptions must serve also to support the atom during its long sojourns in the 1-quantum state. It can only be because hydrogen (and helium) are so light that they are capable of steady-state existence in the high-level chromosphere, under these disadvantages. If they possessed resonance lines in the optical region, they would be capable of existing at much greater heights, and would certainly be found higher than calcium.*

This suggests immediately why the doublets of helium are found extending to a greater height than the singlets. Mitchell found members of the doublet system extending to 7500 kms., whilst members of the singlet system did not extend above 2000 kms. The doublets are derived from the helium atom with coplanar orbits, the singlets from the atom with crossed orbits, and direct radiative transitions between the two are not possible. The lowest level of the doublet system (1σ-level, 4·74 volts), though much higher than the lowest level of the singlet system (oS-level, 24·5 volts), therefore represents a metastable state; it has in many ways the property of a normal state. An atom in the state of coplanar orbits can at most descend, by the emission of lines belonging to the doublet system, to an energy-level of 4·74 volts. Further downward transitions not being directly possible, the atom will in general remain in the 1σ-state until it absorbs another member of the doublet system. Since the sun is relatively rich in radiation of the necessary frequencies, the next absorption will soon occur, and the effect of radiation pressure will be relatively considerable. On the other hand, an atom in the state of crossed orbits may, and eventually will, descend to the oS-level; the level next above this, the 1P-level, is 21·2 volts higher, the lowest frequency capable of effecting a transition is $\lambda 584$, and consequently on the average there will be a long interval before the next absorption. It follows that helium atoms in the state of crossed orbits are relatively far less capable of being supported by radiation pressure, in the solar chromosphere, than atoms in the state of coplanar orbits, and the difference in height between the doublets and the singlets is an immediate consequence.

When atoms in more than two stationary states are present simultaneously, as for hydrogen, a simple extension of the argument used in §§2, 3 does not seem to be possible, and the application of Einstein's radiation theory to chromospheric equilibrium in such cases promises to be rather complicated.

7. *Summary.*—On the assumption that the vapour of ionised calcium in the high-level regions of the solar chromosphere is in a steady state

* Another circumstance helping to increase the effect of radiation pressure on a hydrogen atom is the large number of realisable states of higher quantum number to which the atom has the option of changing by absorption of radiation, as shown by the large number (34) of Balmer lines observed by Mitchell.

supported by radiation pressure, it is shown by applying Einstein's theory of radiation that τ, the average life of an excited Ca^+ atom in the 1π-state, must be less than 5×10^{-8} sec., and the observed intensity of the H and K lines leads to a value $\tau = 0{\cdot}6 \times 10^{-8}$ sec.

A criterion for the permanent existence of a calcium chromosphere is deduced, in the form of an upper limit to the intensity-ratio for the H and K absorption lines, and applied to various stars. It appears probable that the criterion cannot be satisfied for the hotter stars, and consequently ionised calcium atoms may be steadily expelled from them under radiation pressure. Sufficiently cool stars with a high surface value of gravity will be unable to support a calcium chromosphere.

The equilibrium of sodium, hydrogen, and helium atoms in the solar chromosphere is considered in general terms.

The Curvature of de Sitter's Space-Time derived from Globular Clusters. By Ludvik Silberstein.

In a letter written 1924 January 1 to the editor of *Nature*, I derived from de Sitter's cosmology a formula for the Doppler effect $D = d\lambda/\lambda$ of a star in purely inertial relative motion, radial and either from or towards the observer.

This formula practically amounts to

$$D = \pm (1 - \cos^2 \sigma/\gamma^2)^{\frac{1}{2}}, \quad \sigma = r/R \quad . \quad . \quad . \quad (1)$$

where R is the curvature radius of space-time, r the distance of the star at the moment of observation, and γ an integration constant which is but little larger than unity. The upper sign (shift towards the red) corresponds to a receding, and the lower to an approaching star. For distant objects for which $\cos \sigma$ is at all distinguishable from 1, the constant γ may be replaced by its limiting value 1, so that the formula becomes $D = \pm \sin \sigma$, or, to all purposes, for small or moderate σ

$$D = \pm r/R \quad . \quad . \quad . \quad . \quad . \quad (1A)$$

Applying this formula to seven globular clusters for which the distance and the radial velocity are given by Shapley, I find (c being the velocity of light)

	cD. kil./sec.	r. (Astron. units.)	R. (Astron. units.)
N.G.C. 5024	−170	$3{\cdot}8 \times 10^9$	$6{\cdot}7 \times 10^{12}$
5272	−125	2·8	6·7
6205	−300	2·2	2·2
6333	+225	5·0	6·7
6341	−160	2·5	4·7
6934	−350	6·7	5·7
7078	− 95	2·9	9·1
Lesser Magellanic Cloud	+150	4·1	8·2
Greater Magellanic Cloud	+260	3·8	4·4

These give a mean value

$$R = 6{\cdot}0 \times 10^{12} \text{ astronomical units.}$$

As the values of cD are estimated by Shapley to be uncertain to at least ± 25 km./sec., the consistency of these values is good enough, and perhaps better than one is entitled to expect. There are no good estimates of distance for any of the Spiral Nebulæ, but I hope soon to be able to extend this meagre list of globular clusters by two or three more, for which *data* are being obtained by Shapley and others.

It is interesting to notice that the distance of the Great Nebula in Andromeda (cD $= -300$), would by this formula be 6×10^9 astronomical units or 30,000 parsecs, for which various estimates ranging from 2800 to 300,000 parsecs have been given. The Spiral Nebula with greatest radial velocity as yet observed ($+1800$ km./sec.) would be at a distance of $3{\cdot}6 \times 10^{10}$ astronomical units or 180,000 parsecs. This distance is about four times as large as Shapley's estimate of the semi-diameter of our galaxy.

Feeling somewhat confident as to the approximate correctness of

$$R = 6{\cdot}0 \,.\, 10^{12},$$

I have applied this value to a number of special problems. The result of one of these applications, just found, seems interesting enough to be given.

The equation of radial motion of a free particle (star) being, for $\gamma \doteq 1$,

$$\frac{R}{c}\frac{d\sigma}{dt} = \pm \tfrac{1}{2} \sin 2\sigma,$$

the corresponding acceleration is, always apart from gravitation between the observing station and the observed object,

$$\frac{R^2}{c^2}\frac{d^2\sigma}{dt^2} = \tfrac{1}{4} \sin 4\sigma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is, positive, from $\sigma = 0$ to $\frac{\pi}{4}$. For small σ this reduces to $\frac{R^2}{c^2} d^2\sigma/dt^2 = \sigma$ as if the star were "repelled" by the observer.

Now, reinstating gravitation, let $M = c^2L$ be the mass of the particle placed at the origin, and let that of the observed particle be, for simplicity, small as compared with the former. Then the apparent "repulsion" will be counteracted by gravitation which, for small σ, can simply be represented by the Newtonian acceleration $-M/r^2 = -c^2L/R^2\sigma^2$. Thus the equation of radial motion becomes

$$\frac{R^2}{c^2}\frac{d^2\sigma}{dt^2} = \sigma - \frac{L}{R\sigma^2} \quad . \quad . \quad . \quad . \quad (3)$$

This can easily be integrated. But what has seemed particularly interesting was to ask, at what distance $r^* = R\sigma^*$ is the said de Sitterian

"repulsion" just balanced by gravitational attraction? The answer is, by (3),

$$\left(\frac{r^*}{R}\right)^3 = \frac{L}{R}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where L is the gravitation radius of the mass at O, which for (1) (1*a*) played the part of the observing station. If a star be placed at rest just at the distance $r = r^*$, it would remain there. If the star has some initial velocity towards or away from O, the possibilities are somewhat complicated. Yet one may still expect the sphere r^* to be a favoured gathering-place for neighbour stars, and it is therefore that this particular distance as defined by (4) has seemed interesting. An application to the whole galaxy as attracting mass gave a curious result which, however, need not detain us here.

More interesting is the application of formula (4) to our sun. In fact, adopting for the present our preceding value $R = 6{\cdot}0 \cdot 10^{12}$ astronomical units or $9 \cdot 10^{20}$ km., and taking for the sun, in round figures, $L = 1{\cdot}5$ km., I find

$$r^* = 7{\cdot}1 \cdot 10^5 \text{ astronomical units},$$

or the corresponding parallax,

$$p^* = 0''{\cdot}29.$$

Now, among the 19 nearest stars ($p \geq 0''{\cdot}20$) there are as many as *eleven* which are remarkably close to this "neutral" sphere, to wit,

	Parallax.	Distance.
	"	parsecs.
Lacaille 9352	0·29	3·45
Pos. Med. 2164	0·29	3·45
Groomb. 34	0·28	3·57
ϵ Indi	0·28	3·57
O.A.N. 17415	0·27	3·70
Krüger 60	0·26	3·85
ϵ Eridani	0·31	3·23
61 Cygni	0·31	3·23
C.Z. 5^h, 243	0·32	3·12
Procyon	0·32	3·12
τ Ceti	0·33	3·03

Now the number of stars which would fall within these limits out of a number 19 filling uniformly a sphere of 5 parsecs is $19(3{\cdot}85^3 - 3{\cdot}03^3)/125 = 4{\cdot}4$.

Instead of this we have 11, which is a slight crowding round r^*, not very pronounced, but possibly significant enough to support somewhat the value $R = 6{\cdot}0 \times 10^{12}$ derived from the clusters. The values and the signs of the radial velocities of these 11 stars also offer some interesting features.

A refinement of formula (4) by taking into account the finite ratio of both masses offers no difficulties, but will better be postponed to a later communication. Moreover, a modification of L or M of this kind, does not very much influence the value of r^* which is proportional to its cube root.

1924 *January* 18.

Spectroscopic Parallaxes of 100 *B-type Stars.*

By D. L. Edwards, A.R.C.S., D.I.C.

(*Communicated by the Director of the Norman Lockyer Observatory.*)

In an earlier paper on this subject * an attempt was made to show that the absolute magnitude of a B-type star bears well-defined relations to the intensities of certain helium lines in its spectrum, and that the relations are precise enough to afford means of determining luminosities and parallaxes by their aid. The necessary data, in the form of standard parallaxes, were sufficient to establish these relations and to indicate their general nature, though they were scarcely sufficient to define with certainty the exact curves required. Since that time, however, a considerable amount of valuable material has been received, among which may be mentioned Rasmuson's memoir on moving clusters † and a list of trigonometrical parallaxes ‡ of B-type stars from the McCormick Observatory, together with a few trigonometrical parallaxes from various sources. With the aid of this additional material an attempt is made in the present paper to determine these reduction curves finally, and with greater precision than was previously possible. The two pairs of lines, $H_\gamma - \lambda 4388$, and $H_\delta - \lambda 4144$, described in the previous paper, were used, and in addition it was found to be possible to use the pair $H_\gamma - \lambda 4471$. There were thus three independent measures of absolute magnitude for each star, the mean of which may be regarded as satisfactory. The average number of negatives available for each star was three, and two sets of measurements were taken of each negative in order to obtain the best results. The intensity of a line as given by any set of measurements was taken as the mean of about five wedge readings, so that the final value adopted for the intensity of a line represents the mean of about thirty readings. The difference of intensity between the two lines of a pair was plotted against the absolute magnitude of the star as previously described. The three pairs gave curves of a very similar shape, all showing an increase in intensity of the helium lines with increasing absolute magnitude. This increase, which is very rapid for the fainter stars, falls off considerably for the brighter stars (mostly of types B_0 and B_1), after which there appears to be a decrease in intensity, though the material is insufficient to give the precise position of maximum.

* *Monthly Notices, R.A.S.*, **83**, 47.
† *Lund Meddelanden*, Series II., No. 26.
‡ *Publ. A.S.P.*, **34**, 254.

These curves are used in the present paper for the computation of the parallaxes of 100 stars, and the results are compared with those of other workers.

The Reduction Curves.

In the previous work on this subject all the available parallaxes, including trigonometrical, group motion, hypothetical, and others, were used indiscriminately owing to the scarcity of material. For the present purpose, however, in order to obtain trustworthy results, the only parallaxes used in forming the final curves were trigonometrical results by photographic methods and moving-cluster parallaxes (from Rasmuson and Kapteyn). All other data have been used merely as checks on the standard curves, and as a general confirmation of the results. The two classes of parallaxes used have, in the first place, been treated separately in order to test the general agreement of their results, but they have been combined together in forming the final reduction curves.

(*a*) *Trigonometrical Parallaxes.*—These parallaxes have been drawn from the publications of the Allegheny, McCormick, Mount Wilson, Sproul, and Yerkes Observatories, together with some of Gill's parallaxes for Southern stars. Forty-two stars with known parallaxes from these sources were available for measurement, and have been included in the present list. A correction of $+''{\cdot}002$ for Mount Wilson, and $+''{\cdot}005$ for the other observatories, was added to these parallaxes to reduce them from relative to absolute values. Many of these stars have negative parallaxes, but they were all included in order to avoid the systematic error which would result from selecting only those with positive parallaxes. It was therefore necessary to divide these stars into suitable groups, which were formed according to the intensities of their helium lines. For each of these groups the mean parallax was found by giving equal weight to each observatory, and the mean apparent magnitude, $\overline{m}$, was obtained from the formula $\overline{m} = 5 \log \{n/\Sigma 10^{-0{\cdot}2m}\}$, n being the number of stars in the group. The factors $10^{-0{\cdot}2m}$ had previously been computed for other work, and thus were already tabulated and available for use. The mean absolute magnitude was then found and plotted against the mean values of the intensity differences.

(*b*) *Group-Motion Parallaxes.*—The work of Rasmuson, mentioned above, together with that of Kapteyn, which was used in the previous paper, supplied a sufficient number of parallaxes to found reduction curves independently of the trigonometrical values. Those stars for which group-motion parallaxes were available were divided into five groups according to their measured intensity differences, and the mean absolute magnitude of each group found from the individual values of Rasmuson and Kapteyn. The curves obtained in this way agreed extremely well with those derived from trigonometrical parallaxes, especially for the bright stars. In the case of the stars of fainter absolute magnitudes, the trigonometrical curves indicated magnitudes slightly brighter than those given by the group-motion curves, but this difference became inappreciable for the absolutely brighter stars.

In fig. 1 is given a diagram showing the results obtained for one of

the line pairs ($H_\gamma - \lambda 4471$). The broken curve is obtained from trigonometrical parallaxes, and the continuous curve is that founded on group-motion parallaxes. The unit of intensity difference is one-tenth of a wedge division.

(*c*) *The Final Curves.*—These were obtained, as already mentioned, from a combination of the results of trigonometrical and moving-cluster parallaxes. For each of the line pairs under consideration the points

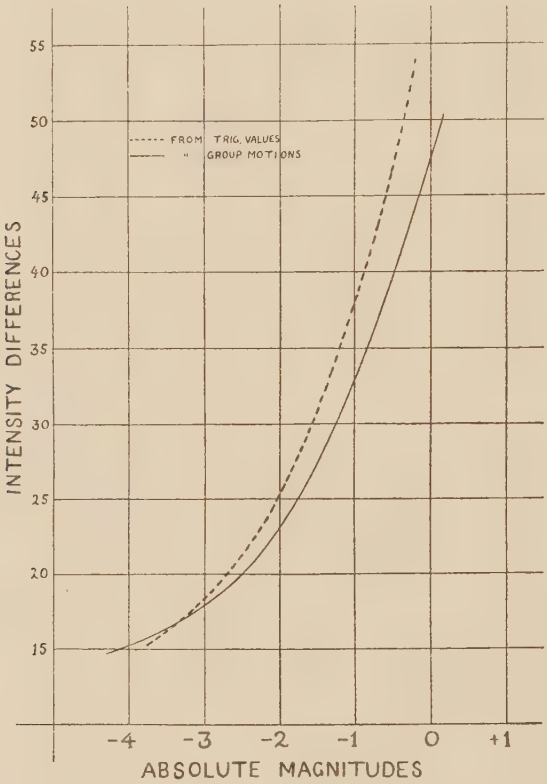


FIG. 1.—Comparison of Reduction Curves based on Trignometrical and Moving-Cluster Parallaxes, for the line pair $H_\gamma - \lambda 4471$.

derived from these two sources were plotted on one diagram, and a mean curve drawn through all the points. This did not differ greatly from either of the individual curves previously described, as will be obvious from fig. 1, and was used for the final determination of parallaxes.

Confirmation from other Sources.

In order to render this investigation as complete as possible, and to check the reduction curves by as many independent methods as were

available, it was decided to use the results of Plummer, Charlier, Van Rhijn, together with those derived from parallactic motion, in a manner similar to that described for the trigonometrical and group-motion parallaxes. Each of the above sources was treated separately, and curves derived expressing the required relation between absolute magnitudes and intensity differences. These were not intended to influence the standard curves, but merely to supply independent confirmation of their general form and accuracy. Differences were found between the check curves and the standards, but it was thought advisable to leave the latter unaltered, since they were based on material more accurate and fundamental than the former.

The assumption made by Plummer* was that the B-type stars are all moving parallel to the galactic plane, and this assumption, combined with radial velocities, gave hypothetical parallaxes for individual stars. These stars were divided into five groups with an average of eight or nine stars to a group, and the mean curves derived from these results showed satisfactory agreement with the standard curves, though some of the individual results as might be expected were widely divergent.

Charlier's work,† implying a relation between absolute magnitudes and type, is interesting in view of the Mount Wilson publication on spectroscopic parallaxes of B stars ‡ which will be discussed later in more detail. The results derived from this source were similar in the mean to those from Plummer's parallaxes, but there was a greater scattering of individual points. The evidence was, however, favourable, and tended to confirm the standard curves.

Van Rhijn's mean parallaxes,§ based on his relation between parallax, proper motion, and apparent magnitude, gave less encouraging results. The 100 stars of Table I. were divided into groups between suitable limits of proper motion and apparent magnitude, and the mean parallaxes found from Van Rhijn's tables. The results gave flatter curves than the standards, and implied a brighter absolute magnitude for the bright stars and slightly fainter magnitudes for the faint stars. Similar results, however, were obtained in the comparison of Van Rhijn's with spectroscopic parallaxes for stars of types K and M, and the divergences mentioned above were not greater than those found for the latter stars.

An attempt was also made to use parallactic motions as a check on the standard curves. The stars were divided into suitable groups, and the mean parallax of the group found from the parallactic motions. Owing, possibly, to the small number of stars in a group, the results were not entirely satisfactory. The divergences were of a similar nature to those found from Van Rhijn's parallaxes, though not so large. The general slope of the standard curves were confirmed, though a difference was indicated of about 0·5 of a magnitude for the brighter stars. This, in the circumstances, can scarcely be regarded as important.

* *Monthly Notices, R.A.S.*, **73**, 174, 450, 492.
† *Lund Meddelanden*, Series II., No 14.
‡ *Ap. J.*, **57**, 294.
§ "Derivation of Change of Colour with Distance, etc.," Gröningen, 1915.

TABLE I.

1. H.D.	2. Name.	3. 1900. R.A.	4. 1900. Dec.	5. m.	6. Spectrum (H.D.).	7. μ.	8. Spectroscopic. M.	9. Spectroscopic. π.	10. Mount Wilson π.	11. Group π.	12 Trig. π
		h m	° ′			″		″			″
886	γ Peg.	0 8·1	+14 38	2·9	B_2	0·013	−2·9	0·007			
2905	κ Cas.	0 27·3	+62 23	4·2	B_0	·011	−3·0	·004			
3360	ζ Cas.	0 31·4	+53 21	3·7	B_3	·023	−2·8	·005			
3901	ξ Cas.	0 36·5	+49 58	4·9	B_3	·021	−1·5	·005			
4727	ν And.	0 44·3	+40 32	4·4	B_3	·028	−1·4	·007			
6811	ϕ And.	1 3·7	+46 42	4·3	B_8	·072	−0·4	·012	...	...	+·009 Ye.
10144	α Eri.	1 34·0	−57 45	0·6	B_5	·093	−1·3	·041	...	...	+·056 Gi.
11415	ϵ Cas.	1 47·2	+63 11	3·4	B_3	·042	−1·8	·009	0″·017	0″·011	−·001 Ye.
16582	δ Cet.	2 34·4	−0 6	4·0	B_2	·010	−3·4	·003			
16908	35 Ari.	2 37·6	+27 17	4·6	B_3	·014	−2·1	·005			
19356	β Per.	3 1·7	+40 34	(2·1)	B_8	·009	+0·4	·045	...	...	+·025 Ye.
20809	− Per.	3 16·1	+48 51	5·3	B_3	·030	−1·3	·005	·007	·009	
21291	− Cam.	3 21·0	+59 36	4·4	B_{9p}	·004	−0·2	·012			
21428	34 Per.	3 22·2	+49 10	4·7	B_5	·044	−1·0	·007	·008	·011	
22928	δ Per.	3 35·8	+47 28	3·1	B_5	·046	−0·5	·019	·020	·011	+·006 M.
22951	40 Per.	3 36·0	+33 39	5·0	B_2	·015	−3·8	·002	·003		
23180	o Per.	3 38·0	+31 58	3·9	B_1	·026	−2·5	·005	...	...	+·005 A, +·035 S.
23302	17 Tau.	3 39·0	+23 48	3·8	B_{5p}	·053	−0·3	·015	·019	·014	
23338	19 Tau.	3 39·3	+24 10	4·4	B_5	·049	−0·5	·011	·013	·013	
23408	20 Tau.	3 39·9	+24 4	4·0	B_5	·054	−0·3	·014	·014	·014	+·012 M, S.
23480	23 Tau.	3 40·4	+23 39	4·3	B_5	·059	−0·4	·012	·012	·015	
23630	η Tau.	3 41·5	+23 48	3·0	B_{5p}	·052	−0·1	·024	·029	·013	
23793	30 Tau.	3 42·8	+10 50	5·0	B_3	·026	−2·9	·003			

24398	ζ Per.	3 47·8	+31 35	2·9	B_1	·021	−2·8	·007			
24760	ε Per.	3 51·1	+39 43	3·0	B_1	·039	−3·3	·005	·010	·009	−·013 M.
25204	λ Tau.	3 55·1	+12 12	(3·4)	B_3	·015	−2·5	·007	...	...	−·007 A.
26912	μ Tau.	4 10·1	+ 8 39	4·3	B_3	·034	−0·9	·009	...	·012	
27396	53 Per.	4 14·3	+46 16	4·9	B_3	·045	−1·6	·005	...	·011	
29248	ν Eri.	4 31·3	− 3 33	4·1	B_2	·002	−3·0	·004	...	·005	
29763	τ Tau.	4 36·2	+22 46	4·3	B_5	·023	−1·3	·008	·010	...	+·012 Ye, +·018 AJ.
30211	μ Eri.	4 40·5	− 3 26	4·2	B_5	·022	−1·1	·009	...	·010	
30836	π^4 Ori.	4 45·9	+ 5 26	3·8	B_3	·007	−3·7	·003	·006	·005	−·007 Ye, +·006 M.
31237	π^5 Ori.	4 49·0	+ 2 17	3·9	B_3	·004	−2·1	·006	·008	·006	+·003 M.
32630	η Aur.	4 59·5	+41 6	3·3	B_3	·081	−1·7	·010	·011	·020	+·017 M.
33328	λ Eri.	5 4·4	− 8 53	4·3	B_2	·008	−1·7	·006	...	·007	
35039	22 Ori.	5 16·7	− 0 29	4·7	B_3	·005	−2·6	·004	·005	·005	
35149	23 Ori.	5 17·6	+ 3 27	5·0	B_3	·007	−1·7	·005	...	·005	
35411	η Ori.	5 19·4	− 2 29	3·4	B_1	·006	−2·7	·006	·007	·006	
35468	γ Ori.	5 19·8	+ 6 16	1·7	B_2	·020	−2·5	·014	·018	·009	+·029 M.
35497	β Tau.	5 20·0	+28 31	1·8	B_8	·180	−0·5	·034	·040	...	+·035 M, +·010 Ye.
35715	ψ Ori.	5 21·6	+ 3 0	4·7	B_2	·014	−2·7	·003	·008	·006	−·010 Ye.
36267	32 Ori.	5 25·4	+ 5 52	4·3	B_3	·038	−0·5	·011	...	·016	
36486	δ Ori.	5 26·9	− 0 22	2·5	B_0	·003	−3·1	·008	·010	·005	+·011 M, −·018 AJ.
36822	φ′ Ori.	5 29·3	+ 9 25	4·5	B_0	·008	−1·7	·006	...	·007	·000 M.
37128	ε Ori.	5 31·1	− 1 16	1·8	B_0	·002	−3·1	·011	·014	·005	·008 AJ.
37202	ζ Tau.	5 31·7	+21 5	3·0	B_{3p}	·028	−1·9	·011	·019	·010	
37438	125 Tau.	5 33·5	+25 50	5·0	B_3	·044	−2·1	·004			
37468	σ Ori.	5 33·7	− 2 39	3·8	B_0	·001	−3·0	·004	·005	·005	
37742	ζ Ori.	5 35·7	− 2 0	2·1	B_0	·007	−3·2	·009	·011	·006	−·020 AJ.
37795	α Col.	5 36·0	−34 8	2·8	B_{5p}	·035	0·0	·027			

TABLE I.—*continued.*

1. H.D.	2. Name.	3. 1900. R.A.	4. 1900. Dec.	5. m.	6. Spectrum (H.D.).	7. μ.	8. Spectroscopic. M.	9. Spectroscopic. π.	10. Mount Wilson π.	11. Group π	12. Trig. π.
		h m	° ′			″		″	″	″	″
38771	κ Ori.	5 43·0	− 9 42	2·2	B_0	0·006	−3·7	0·007	0·012	0·006	+·029 AJ.
40111	139 Tau.	5 51·8	+25 57	4·9	B_2	·004	−2·2	·004	...	·005	
42560	ξ Ori.	6 6·3	+14 14	4·4	B_3	·035	−1·5	·007	...	·013	
44743	β CMa.	6 18·3	−17 54	2·0	B_1	·007	−3·8	·007	·014	·007	+·012 M.
52089	ϵ CMa.	6 54·7	−28 50	1·6	B_1	·001	−4·4	·006	...	·006	
74280	η Hya.	8 38·0	+ 3 46	4·3	B_3	·022	−1·5	·007	...	·008	
90994	30 Sex.	10 25·2	− 0 8	5·0	B_5	·054	+0·1	·011	·006		
91316	ρ Leo.	10 27·5	+ 9 49	3·9	B_{0p}	·010	−2·9	·004	·006	...	+·031 M.
92938	− Car.	10 38·7	−63 57	5·2	B_3	·018	−2·0	·004			
93030	θ Car.	10 39·4	−62 52	3·0	B_0	·024	−3·7	·004	...	·009	
93194	− Car.	10 40·5	−63 26	5·1	B_3	·014	−1·5	·005			
93607	− Car.	10 43·2	−63 52	5·1	B_5	·014	−2·8	·003			
100600	90 Leo.	11 29·5	+17 21	5·8	B_3	·011	−1·8	·003	·005		
109387	κ Dra.	12 29·2	+70 20	3·9	B_{5p}	·060	−0·6	·013	...	...	−·002 M.
110879	β Mus.	12 40·1	−67 34	3·3	B_3	·041	−2·4	·007	...	·013	
111123	β Cru.	12 41·9	−59 9	1·5	B_1	·055	−3·3	·011	...	·018	+·013 Gi.
116658	α Vir.	13 19·9	−10 38	1·2	B_2	·052	−2·1	·022	·038	...	+·014 M, −·007 Gi.
118716	ϵ Cen.	13 33·5	−52 57	2·6	B_1	·041	−2·9	·008	...	·012	
120315	η UMa.	13 43·6	+49 49	1·9	B_3	·119	−0·8	·028	...	...	+·006 M.
122451	β Cen.	13 56·8	−59 53	0·9	B_1	·041	−3·2	·015	...	·012	+·042 Gi.
129056	α Lup.	14 35·3	−46 58	2·9	B_2	·036	−2·0	·011	...	·009	
132058	β Lup.	14 52·0	−42 44	2·8	B_{2p}	·070	−2·4	·009	...	·018	
132200	κ Cen.	14 52·7	−41 42	3·4	B_3	·036	−1·9	·009	...	·009	

138690	γ Lup.	15 28·5	−40 50	3·0	B_3	·039	−1·8	·011	...	·010	
143018	π Sco.	15 52·8	−25 50	3·0	B_2	·039	−1·4	·014	...	·010	
144217	β Sco.	15 59·6	−19 32	2·9	B_1	·031	−3·0	·007	·007	·009	−·019 AJ.
147165	σ Sco.	16 15·1	−25 21	3·1	B_1	·034	−1·5	·012	...	·009	
147394	τ Her.	16 16·7	+46 33	3·9	B_5	·032	−0·5	·013			
149438	τ Sco.	16 29·7	−28 1	2·9	B_0	·038	−1·8	·012	...	·010	
151890	μ^1 Sco.	16 45·1	−37 53	3·1	B_{3p}	·030	−1·9	·010	...	·008	
151985	μ^2 Sco.	16 45·6	−37 51	3·6	B_2	·032	−3·5	·004	...	·008	
157056	θ Oph.	17 15·9	−24 54	3·4	B_3	·031	−3·2	·005	...	·008	
157246	γ Ara.	17 17·0	−56 17	3·5	B_1	·013	−3·9	·003	...	·003	
158148	− Her.	17 22·5	+20 10	5·4	B_5	·016	−0·4	·007	·007		
158408	υ Sco.	17 24·0	−37 13	2·8	B_3	·042	−2·4	·009	...	·011	
158926	λ Sco.	17 26·8	−37 2	1·7	B_2	·036	−2·9	·012	...	·009	
160578	κ Sco.	17 35·6	−38 59	2·5	B_2	·028	−3·2	·007	...	·007	
160762	ι Her.	17 36·6	+46 4	3·8	B_3	·008	−2·6	·005			
164284	66 Oph.	17 55·3	+ 4 23	4·8	B_3	·019	−0·7	·008	...	·007	
164353	67 Oph.	17 55·6	+ 2 56	3·9	B_{5p}	·014	−2·2	·006	·007	·005	
164852	96 Her.	17 58·1	+20 50	5·1	B_3	·018	−3·1	·002	·005	...	+·001 Ye.
193322	− Cyg.	20 14·6	+40 25	5·8	B_{2p}	·012	−1·3	·004	·003	...	+·008 W.
195810	ϵ Del	20 28·4	+10 58	4·0	B_5	·027	−1·1	·010	...	...	+·018 Ye, −·004 M.
198183	λ Cyg.	20 43·5	+36 7	4·5	B_5	·011	−0·1	·012	...	...	+·028 S.
200595	− Cyg.	20 59·3	+45 27	6·2	B_8	...	−0·9	·004	...	...	+·001 A.
205021	β Cep.	21 27·4	+70 7	3·3	B_1	·012	−3·0	·005	·008	...	−·011 Ye, +·019 M.
209952	α Gru.	22 1·9	−47 27	2·2	B_5	·200	+0·2	·040	...	...	+·029 Gi.
212120	2 Lac.	22 16·9	+46 2	4·7	B_5	·020	−0·2	·011	...	...	+·041 S.
217675	o And.	22 57·3	+41 47	3·6	B_5	·032	−0·9	·013	·019	...	+·004 M.
223229	− And.	23 42·6	+46 16	5·8	B_3	...	−2·0	·003			

In fig. 2 the results from the various confirmatory data for one of the line pairs measured ($H_\gamma - \lambda 4471$) are brought together in one diagram. The standard curve for this pair is included, and it will be seen that there is sufficiently good agreement to afford additional confidence in the use of this curve for determining parallaxes. This

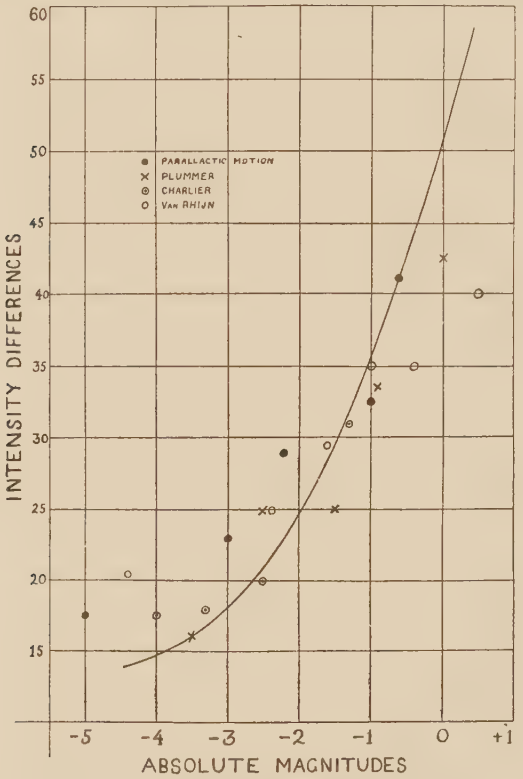


FIG. 2.—Comparison of Standard Curve for the pair $H_\gamma - \lambda 4471$ with other Data.

diagram may be taken as typical of the results for all the reduction curves that have been employed.

The Parallaxes.

In Table I. are given the parallaxes of 100 stars as finally determined from the reduction curves described above. Most of the stars in the earlier list have been included, since the reduction curves as now defined vary slightly from those originally used. Also, it was thought advisable to remeasure these stars in order to test the reliability of the intensity measures. Slight variations were found, but they were rarely of much

importance, so that differences between parallaxes in this and the previous list arise mainly from errors in the reduction curves originally employed.

In column 1 is given the number of the star in the Henry Draper Catalogue. Columns 2, 3, and 4 give the name or constellation and approximate position of the star for 1900. Column 5 gives the visual magnitude, 6 the spectral type (H.D.), and 7 the total proper motion. Columns 8 and 9 give the spectroscopic absolute magnitude and parallax, and the remaining columns give other parallaxes for comparison. The letters following trigonometrical parallaxes in the final column indicate the authorities, and have the following significations:—A = Allegheny, M = McCormick, Gi = Gill, S = Sproul, W = Mount Wilson, Ye = Yerkes, AJ = parallaxes quoted by Adams and Joy in their paper on B stars (*loc. cit.*), but which could not be traced elsewhere.

Comparison of the Results with Other Parallaxes.

The agreement of the results given in Table I. with the standard trigonometrical and moving-cluster parallaxes are, as might be expected, very close. The mean trigonometrical parallax of those stars which have such values is +0″·012 and the mean spectroscopic value for the same stars is 0″·013. For those stars belonging to moving clusters the mean group parallax is 0″·0090, the mean spectroscopic value being 0″·0085.

The relation between the present results and those of Plummer, Charlier, and Van Rhijn have been sufficiently indicated in a previous paragraph, and in the diagram given in fig. 2. There are some large individual discordances, but the general agreement is sufficiently good to strengthen the position of the spectroscopic parallaxes.

A comparison was also made with "dynamical" parallaxes (based on the orbital motion of binaries), and the stars in the above list for which parallaxes of this nature were known are brought together in Table II. Owing to the scarcity of these parallaxes for B stars, no attempt has been made to exhibit the results in diagrammatic form as was done for the other comparisons.

TABLE II.

Comparison of Results with "Dynamical" Parallaxes.

H.D.	Star.	*m.*	Type.	M.		π.	
				Spec.	Dyn.	Spec.	Dyn.
6811	ϕ And.	4·3	B_8	−0·4	−0·9	·012	·009
19356	β Per.	...	B_8	+0·4	−0·8	·045	·026
25204	λ Tau.	...	B_3	−2·5	−3·1	·007	·005
36267	32 Ori.	4·3	B_3	−0·5	+0·7	·011	·019
37468	σ Ori.	3·8	B_0	−3·0	−1·2	·004	·010
37742	ζ Ori.	3·8	B_0	−3·2	−1·9	·009	·016
138690	γ Lupi.	3·0	B_3	−1·8	+0·3	·011	·028
198183	λ Cyg.	4·5	B_5	−0·1	+1·0	·012	·020
223229	HR 9011	5·8	B_3	−2·0	+1·2	·003	·012
Pleiades (5 stars)		...	...	−0·3	−1·1	·015	·010

The parallaxes in the last column are taken mainly from Jackson and Furner's list * and from Shapley's memoir on eclipsing binaries.† The parallax of the Pleiades is the mean value found by Trumpler. Apart from the rather large individual differences shown in this table, there seems to be a systematic difference of about 0·5 of a magnitude between the mean values of the spectroscopic and dynamical absolute magnitudes, the former being brighter. This would appear to imply an error in the average mass of B stars used for computing the above parallaxes, but the material is too scanty to afford very precise information on this subject.

The results of Adams and Joy, which are given in column 10 of Table I., are based on an accurate classification of spectral types, and afford an instructive comparison with the present results, since, although both are spectroscopic, there is considerable difference between the two methods. Thirty-seven stars of Table I. have spectroscopic parallaxes by Adams and Joy, and these show a distinct difference between the results obtained by the two methods, since the magnitudes of column 8 are, with only three exceptions, consistently brighter than those of Adams and Joy, the average difference being 0·6 of a magnitude. In order to investigate the reason for this difference, an examination was made of those stars measured by Adams and Joy which also possess trigonometric parallaxes. The mean trigonometric parallax of these stars (27 in number) is 0″·007. The mean value from column 9 is 0″·011, and from Adams and Joy (column 10) 0″·015. These figures, therefore, support the present results rather than those of Mount Wilson, and even indicate a larger divergence from the latter than that found previously. Similar confirmation is given by a consideration of group parallaxes, and an examination of the results common to the three methods gives a mean value of 0″·009 for the group parallaxes, 0″·009 for the present results, and 0″·012 for Mount Wilson. These considerations, taken in conjunction with the good agreement of the general mean of the trigonometric and group parallaxes as found previously, seem to give support to the method employed in the present paper.

The Mount Wilson values are based on the assumption of a relation between spectral type and absolute magnitude. This relation is strongly supported by the results of the present paper, and it may be possible that the measurement of the intensities of the helium lines simply amounts to an accurate method of subdividing the spectral types. It would appear, however, that there is a considerable range of absolute magnitude for any one type, though this range is reduced when Adams and Joy's division into sharp and diffuse line stars is taken into account. In Table III. is given a comparison of the mean absolute magnitude (as here derived) for Adams and Joy's spectral types with the mean values which they used in deducing parallaxes. The range of magnitude covered by each type is indicated as far as the material will allow, but the scarcity of stars in some of the types renders this somewhat ambiguous. The mean absolute magnitude for the Harvard (H.D.) types are also

* *Monthly Notices, R.A.S.*, 81, 2; *ibid.*, 83, 4.
† *Contrib. from Princeton Obs.*, No. 3.

added, and the relation is thus well shown between spectral type and absolute magnitude.

TABLE III.

M.W. Type.	No. of Stars.	Mean M (N.L.O.).	Range of M.	Mean M (M.W.).	Harvard Type.	Harvard Mean M.
B_{0n}	2	−3·1	0·2	−3·1	B_0	−3·2
B_1	6	−3·3	1·1	−2·6		
B_{1n}	4	−3·3	0·6	−2·4	B_1	−3·1
B_{2s}	3	−2·4	0·4	−2·0		
B_{2n}	1	−1·3	...	−1·5	B_2	−2·6
B_{3s}	3	−2·3	1·4	−1·5		
B_{3n}		−1·9	1·7	−0·9	B_3	−1·9
B_{4n}	2	−1·6	0·6	−0·6		
B_{5s}	1	+0·1	...	−1·1	B_5	−0·7
B_{5n}	5	−0·9	1·4	−0·5		
B_{7n}	2	−0·9	...	−0·1	B_8	−0·3
B_{8n}	2	−0·2	0·2	+0·1		
B_{9s}	2	−0·4	0·2	−0·2	B_9	−0·2

The Mount Wilson division into sharp and diffuse line stars is also supported by these results. The mean absolute magnitude deduced for the 15 stars of the Mount Wilson "*s*" class is −2·3, and for the 21 "*n*" stars is −1·8; thus indicating that the "*s*" stars are 0·5 of a magnitude brighter, in the mean, than the "*n*" stars—a result which is in good agreement with that found at Mount Wilson.

Summary.

This paper is the continuation of a previous one on the same subject.

With the aid of better and more recent data, the reduction curves, giving the relation between absolute magnitudes and intensities of certain helium lines, are more accurately defined, and are checked by a comparison with other material.

The helium lines λ4471, λ4388, and λ4144 are used, and compared in intensity with the hydrogen lines H_γ and H_δ.

The curves are used for computing the parallaxes of 100 B stars, and these results are discussed and compared with those of other workers.

On Editing Newton. By Professor R. A. Sampson, F.R.S.

The valuable address of the President at the Annual Meeting recalls a project with the same intention of exactly twenty years ago, of which I may take leave to speak, for I occupied a prominent place in it, though, as events proved, not very usefully. Omitting the ancillary steps of the story, the Cambridge University Press at that time agreed to produce a complete edition of Newton's Scientific Works and Correspondence, to be prepared under the auspices of the Cambridge Philosophical Society, which was represented by a strong editorial committee of residents, while I was charged with the duties of Editor-in-Chief. The Syndics of the Press undertook the whole financial responsibility, including the provision of expert clerical work in transcribing MSS., etc., as well as necessary personal expenses of the editor. Had the manuscript ever been produced, the work might before now have been accomplished.

It may be useful if I give here the actual outlines of the edition proposed, which was accepted by the committee of the C.P.S. and by the Syndics.

Outline of Proposed Critical Edition of Newton's Scientific Works and Correspondence.

Scope. The following proposals embrace the whole of the scientific works edited from the originals as far as these are accessible, including all letters and any fragments of interest that are among the Portsmouth papers or elsewhere; they also include an English edition of the *Principia* in addition to the Latin original.

Form. The form of issue may be left open until some experimental pages have been set up by the Press, but to supply a definite idea of the extent of the work the following estimates are based on the supposition that a small quarto page is adopted containing some 320 words, with figures in the text.

[If the figures are inserted in their proper places in the midst of the text with the repetitions that this entails, an addition to the estimated pages must be made. I have measured up the figures approximately, and give in brackets the additional pages.—R.A.S.]

Editorial matter. The text to be accompanied by marginal analysis, recension of textual changes where these occur, notes mainly bibliographical and historical, and full index. This additional matter has been estimated to occupy about one-tenth of the space of the text.

Extent and time. I believe the following estimates of the extent of the work may be relied upon, that is, the whole work would be covered by six small quarto volumes of about 600 pages each.

It would necessarily take a considerable time to complete—say six years for the six volumes.

Expenses. Some expenses besides those of printing and publishing

must be provided for. There will be a large amount of collating, which would probably be best done by the occasional services of a clerk expert at such work.

Principia reprinted from Latin of Edition III., with bibliography, recension of textual variations, notes of brief extent, and full index, as mentioned above.

In small quarto, with 350 words to the page and figures in the text as printed, the work occupies 530 pages. If we take 320 words and figures in the text—there are 282 figures—and add one-tenth for notes, etc., the whole would be embraced by about 600 pages.

[Additional pages if figures are among the text (62).]

(1) *Opticks* from 3rd (or 4th) English edition with recension as above. The book contains about 100,000 words and would occupy 300 pages.

(17 pp.) [Add for figures (17).]

(2) *Analysis per Æquationes Numero Terminorum Infinitas* (1669).

This might be taken from the copy printed by Jones in 1711.

(2 pp.) [Figures (2 pp.).]

(3) *Methodus Fluxionum et Serieum Infinitarum*, given by Horsley under the name *Geometria Analytica sive Specimina Artis Analyticæ* (written before 1671, first published in English by Colson, 1736—it may be printed from Horsley, who worked from MSS. of Newton's).

(9 pp.) [Figures (9 pp.).]

(4) *De Quadratura Curvarum.* First printed with the *Opticks* of 1704.

(1 p.) [Figure (1 p.).]

(5) *Methodus Differentialis.* First published by Jones from Newton's MS. in 1711.

(6) *Enumeratio Linearum Tertii Ordinis.* First published with the *Opticks* in 1704.

(13 pp.) [Figures (13 pp.).]

Allow for notes, etc., 30 pages.

(+42 pp.) [There are MSS. of several of these, which may or may not be complete, among the Portsmouth papers; as I have not examined these, I have mentioned above what seems the most authoritative of the published copies.]

Containing Lectures delivered as Lucasian Professor.

(1) *Lectiones Opticæ*, delivered in 1669, 1670, 1671, 1672. There is a MS. of these in the University Library with which printed copies could be collated.

(27 pp.) [Figures (27 pp.).]

(2) *Arithmetica Universalis*, delivered 1673–4–5–6–7–8–9–80–1–2–3.

A MS. of these in the University Library for correction of printed edition.

(10 pp.) [Figures (10 pp.).]

60 pp. (3) *De Motu Corporum*, delivered 1684–1685. MS. of these in University Library.

(6 pp.) [Figures (say 6 pp.).]

60 pp. (4) *De Motu Corporum Liber*, commonly called *De Mundi Systemate*, delivered 1687– . MS. in Trinity College Library, from which printed copies may be corrected. It contains some 20,000 words.

(3 pp.) [Figures (3 pp.).]

40 pp. Allow for notes, etc., 40 pages.

600 pp. (+46 pp.) Letters and miscellaneous pieces.

Vols. IV., V. These would include the communications to the Royal Society, which were mainly in the form of letters to Oldenburg, the Secretary.

Also the *Commercium Epistolicum.*

Also fragments from the Portsmouth MSS. or elsewhere that seemed to deserve publication.

In many cases it would be necessary to print the letter of Newton's correspondent as well as Newton's reply.

It is not possible to give definite statements as to the extent of these, even where the material is already published, for it is printed in such different scales and is so scattered that it is hard to collect it into a sum, but as far as I can estimate, the papers already published would cover between 800 and 900 pages, allowing about 300 pages for fresh material and notes.

The sources are chiefly the following:—

(1) Letter books of the Royal Society.

(2) The Portsmouth Papers.

110 pp. The most famous letters in these two collections have been published by Horsley and Brewster. They occupy some 40–50 pages of the former and 60–80 of the latter.

156 pp. (3) The Macclesfield Correspondence. Newton's letters occupy 156 pages.

286 pp. (4) Correspondence with Cotes. With other letters printed by Edleston this fills 316 pages, of which *some* 30 *need not be reproduced* [as they are not Newton's and do not relate very directly to him].

100 pp. (5) Correspondence with Flamsteed. There are 30–40 letters on each side.

13 pp. (6) Four letters to Dr. Bentley; these occupy 13 pages in Horsley.

10 pp. (7) Thirteen letters to Dr. Covell, say 10 pages.

(8) Letters in British Museum; some of these are printed by Edleston.

(9) There is a limited number of other letters in college libraries, at Christ's Hospital and elsewhere, which could not greatly affect the bulk.

120 pp. (10) *Commercium Epistolicum.* 120 pages in Horsley.

20 pp. (11) Minor papers in *Phil. Trans.*, solutions of problems, etc.

385 pp. Allow for new material, 385 pages.

1200 pp.

Supplementary Volume: The Principia in English. It was, I think, the opinion of the committee that an edition of the *Principia* in English, which might be purchased independently of the rest, would be a work of real and wide usefulness. There is a contemporary translation by A. Motte (1729) which is in itself a good piece of work and has been thrice reprinted. We should revise Motte's translation minutely to gain more of the precision of Newton's own words, and add notes, running analysis in margin, and index, as in the other volumes. Motte's translation contains some 220,000 words. If we add one-tenth for notes, etc., and if the plates were *on inserted slips*, in order that it might be put in one convenient volume, it would be necessary to print with different types from the rest of the edition, namely about 400 words to the page.

[With the same type as the rest of the edition it would occupy 750 pages.

Additional for figures among the text (62 pp.).]

R. A. Sampson.

1904 *April* 18.

It still remains a matter of lively regret to me that the work was never carried out. On reading through the scheme again, after the lapse of several years, it appears a very attractive one. My first feeling is one of some surprise that with so prosperous an opening I let it go. An explanation is due to those who were more than willing to collaborate in the production. This may be given very briefly. Perhaps I ought not to have undertaken it. In 1904, when the scheme was approved, I had upon my hands a very heavy piece of work, the Theory and Tables of Jupiter's Satellites, which I might never have finished had I allowed any other task to take priority of it. As often happens, it took more time than was anticipated. The conclusion was assured in 1910; and, had nothing intervened, I should then have taken up actively the edition of Newton, though in the meantime I had learned to think rather temperately of my own qualifications for the work. But in the same year I was transferred to Edinburgh where different demands upon my time have never allowed me to think of it again. As a matter of interest, I may now add a few remarks which review and recollection suggest. These do not all point in one direction.

The purpose, I presume, is to place in the hands of scholars a critical text of what Newton actually wrote, complete at least in its scientific bearings. It would be but little read by ordinary students and investigators. It is an historical task, and, in a sense, one of national importance, but for the advancement of science it matters only in moderate degree. On the other hand, no one could presume to handle it who was not completely versed in the latest developments of the subjects Newton treated. His words do not age. One example will show what I mean. Newton attempted the Lunar Theory in Book III. of the *Principia*. With that faultless eye for mathematical form, which has made him perhaps as great in pure mathematics as in applied, he based his discussion upon the Variational Orbit, which he invented

or discovered, deducing theoretically the sizes of the greatest and least axes. The significance of this might easily escape one who did not remember how Hill's recognition of the same orbit and his generalisation of it form the foundation of all we now know about Periodic Orbits. For another example, there are the philosophical passages on Space and Time, the interest of which is enhanced rather than diminished if we are now offered a coherent alternative to them. One is reminded of such connections and suggestions *passim*, and, personally, I am reminded how little I know about them. However brief the notes to an edition might be, it is essential that the significance of all such passages should be correctly indicated. Examined under the microscope, Newton's work shows no cracks or crevices. In consequence, everyone who has worked on Newton has fallen under a spell. Except, perhaps, Horsley; but the ineptitude of Horsley, who included his own lucubrations under the same covers and on a par with Newton's writings, needs no underlining. Dr. Dreyer has remarked that Newton has fallen into the hands of hero-worshippers.* It is quite true. That proud, reserved, commanding face that speaks from the portrait in our rooms laid a spell on his contemporaries. His writings for a century after laid a spell on British mathematicians, who, however, flung away the better half of it, and lived contented with the worser part. The spell lasts, in some degree, to this day. It is not easy, for instance, to resign oneself to the statement of the syndicate who examined the Portsmouth Papers, that Newton's writings on Alchemy contain "very little of interest in themselves," knowing how much thought and labour he concentrated on the matter.

Herein lurks a considerable risk that another bad edition of Newton's works might be produced, in the sense of reading too much into it. It demands judgment as much as knowledge. Moreover, as already remarked, it is an historical task rather than a scientific one. Before anyone undertakes it he should see clearly that he is making a voyage into the past, the greater part of it completely dead and obsolete. One must not modernise, but before the argument can be seized, even the notation often requires to be construed, like the Latin in which all, except the letters and the *Opticks*, are written. An editor must allow himself to be absorbed into this defunct world for many years. Probably he would never quite find his way out of it. Trying to reconstruct the relations of Newton with Flamsteed and Halley, or the personalities concerned in the *Commercium Epistolicum*, is an intriguing occupation. Though the greater portion of all the documents is already in print, and in many parts has been edited with punctilious care, the purely historical problem is full of puzzling discriminations; and these are necessary if the new edition is to present these celebrated causes as in a final Court of Appeal. But from a modern point of view, one may say of them, as Carlyle said of the plot of "The Ring and the Book"—they only want forgetting.

And there is nothing very new to say. The monographs and chapters dealing with Newton's significance in special bearings are legion; compare, for example, the very full and careful treatment on the

* *History of the Royal Astronomical Society*, p. 78.

mathematical side in Cantor's *History*, and in Zeuthen's *History of Mathematics in the Sixteenth and Seventeenth Centuries.* All these scattered writings would require to be considered and their outcome re-issued.

Without overrating what one can do in the modern world and the modern problems of science, one might well hesitate before advising anyone except a professional historian to tie himself to such a piece of work; and a professional historian would be only too likely to feel a merely tepid interest in such a statement as "a ray of light has four sides."

Generally, I feel that the task is one that only a fairly senior man should engage upon; one who knows what to expect of himself, and who brings judgments already formed on numberless questions of inclusion and exclusion, and on the general bearing and importance of the topics that arise.

Another point I mention after some hesitation. Scientific men are used to foregoing any direct payment for their work; and indirect payment by credit may come generally to the same thing. But there are limits. In this case, especially if the editor were a member of a committee which might include men much more eminent than himself, he would run a risk, that should be clearly contemplated, of spending his years and labour and getting nothing at all, not even credit. A senior man would see this, and would take the work or leave it, with his eyes open, according to his point of view. A junior ought not to be encouraged to speculate in his future to that degree.

The Metonic and Callippic Cycles. By J. K. Fotheringham.

The first Miletus calendar fragment, No. 84 (*Sitzungsberichte der königl.-preussischen Akad. der W.*, Berlin, 1904, 1, pp. 92 ff., 266 ff., 752 ff.), contains the notice:

"From the summer solstice of Scirophorion 13 under Apseudes, being Phamenoth 21 Egyptian, to the summer solstice of Scirophorion 14 under Polyclitus, being Payni 11 Egyptian, and in the Milesian (calendar). . . ."

The archonship of Apseudes is known to have been 433–2 B.C. In that year Phamenoth 21 fell on June 27 Julian, and the summer solstice in the morning of June 28 Julian. From this we learn that the summer solstice was observed on June 27 of that year, and that that day was called, it is not stated by whom, Scirophorion 13.

Diodorus, 12, 36, has: "In the archonship at Athens of Apseudes the Romans appointed as consuls Titus Menenius and Proculus Geganius Macerinus. . . . And at Athens Meton, the son of Pausanias, being famous in astronomy, published the well-known 19 years' cycle, making it begin from the thirteenth day of the month Scirophorion at Athens. . . ."

From this it is clear that Scirophorion 13 in the specified year

was the initial date of Meton's cycle, from which it would appear that the cycle was made to begin on the day of the summer solstice and that this day was called Scirophorion 13 in the Metonic cycle. It is far more likely that Diodorus is giving us the initial date shown on the calendar than the corresponding date in the civil calendar. This is, of course, confirmed by the comparison of the two solstice observations on the calendar of Miletus, seventeen Metonic cycles apart. In testing the accuracy of the Metonic cycle by these observations it was natural to use the Metonic date. The civil calendar would afford no evidence of the age of the moon.

Lastly Ptolemy, *Meg. Syn.*, 3, 1 (p. 205, Heiberg), writes: "And if, because of its antiquity, we compare the summer solstice observed by Meton and Euctemon and their school, but somewhat roughly recorded, with the one which we have reduced with the utmost accuracy, we shall find the same result. For that is recorded to have happened in the archonship of Apseudes at Athens on Phamenoth 21 Egyptian in the early morning. . . ."

This makes it clear that the solstice, which, as we have seen, served as the initial day of Meton's cycle, was observed by Meton and Euctemon in the early morning (πρωΐας) of June 27 Julian in 432 B.C. As this was approximately the 13th day of a lunar month, everything is in agreement. It is strange that, in the face of the direct statement in Diodorus, modern scholars should almost unanimously have supposed that the cycle was regarded as beginning on Hecatombæon 1 and not on Scirophorion 13. Some older scholars, such as Scaliger, *De Emendatione Temporum* (1629), pp. 45, 46, and Dodwell, *De Veteribus Græcorum Romanorumque Cyclis* (1701), pp. 175, 176, ventured to suppose that the cycle began on Scirophorion 13 of the civil calendar of the time, which they equated with the Metonic Hecatombæon 1. Petavius, *De Doctrina Temporum*, 50, ii., c. x., tom. i. (1703) p. 57, appears to be alone in recognising that Scirophorion 13 is the Metonic date of the beginning of the cycle.

Details of the calendar without reference to the initial date are given by Geminus, 8, 50 ff.: "So, when the octaeteris proved to be altogether erroneous, the astronomers of the school of Euctemon, Philippus, and Callippus devised another period. For they observed that in 19 years there are contained 6940 days, and 235 months, including intercalary months, and in the 19 years there are 7 intercalary months. The year, then, according to them, consists of $365\frac{5}{19}$ days. And of the 235 months they made 110 hollow and 125 full, so that instead of their being alternately hollow and full, there were sometimes two full months in succession. For this is the natural result of the phenomena considered in relation to the moon, though it was not regarded in the octaeteris. And their reason for making 110 of the 235 months hollow was as follows: Since there are 235 months in the 19 years, they made a presupposition that these were all months of 30 days. Then the days amount to 7050 altogether; but 110 ought to be regarded as hollow, wherefore the actual number of days in the 19 years' cycle, according to the moon, comes to 6940. So if all the months are of 30 days, the 7050 days exceed the 6940 days by 110.

Therefore they treat 110 months as hollow, in order that the 6940 days of the 19 years' cycle may be completed in the 235 months. In order that the elimination of days might take place as far as possible at equal intervals, they divided the 6940 days by 110. This gives 63 days. Accordingly after every 63 days it is necessary to eliminate a day in this cycle. And it is not always the 30th day that is to be eliminated, but the day that falls after an interval of 63 days is regarded as the day to be eliminated."

This gives us a clear rule for the lengths of the months, but no rule for the distribution of intercalary months, nor for the placing of the five intercalary days required to bring the 19 solar years to the required length. Curiously enough Scaliger, *op. cit.*, p. 78, and Dodwell, *op. cit.*, pp. 50–52, alone among modern scholars, have been willing to accept Geminus's express statement that every 64th day was omitted, irrespective of its position in the month.* I see no reason to dispute this statement, which gives us the means of marking off the days of the month in this cycle, though not of identifying those months. In the following table I give the date of the first day of each month in the first year of the Metonic cycle, to which I have added the Athens mean time of mean new moon. As Meton knew nothing of the inequalities in the motion of the moon, I have thought it superfluous to give the time of true new moon.

Number of Month.	Days elapsed before Beginning of Month.	Julian Date of First Day.	Mean New Moon.
1	18	July 15	July 15, 12.52 p.m.
2	48	Aug. 14	Aug. 14, 1.36 a.m.
3	78 – 1	Sept. 12	Sept. 12, 2.20 p.m.
4	108 – 1	Oct. 12	Oct. 12, 3.4 a.m.
5	138 – 2	Nov. 10	Nov. 10, 3.48 p.m.
6	168 – 2	Dec. 10	Dec. 10, 4.32 a.m.
7	198 – 3	Jan. 8	Jan. 8, 5.16 p.m.
8	228 – 3	Feb. 7	Feb. 7, 6.0 a.m.
9	258 – 4	Mar. 8	Mar. 8, 6.44 p.m.
10	288 – 4	Apr. 7	Apr. 7, 7.29 a.m.
11	318 – 4	May 7	May 6, 8.13 p.m.
12	348 – 5	June 5	June 5, 8.57 a.m.

It will be observed that if each day is reckoned as beginning at 6 p.m., or mean sunset on the day before that shown in my table, mean new moon will in each of these months except the ninth fall on the first day of the Metonic month. Now the official civil day at Athens began at sunset, and, as seen above, the first day of Meton's cycle was the day of the summer solstice. So it would appear that his design was that his years should begin at the sunset before the summer solstice and his months at the sunset before each mean new moon.

It is nowhere stated which of his solar years were made to include

* Dodwell understands Geminus to mean every 63rd day.

366 days. The solar years were reckoned by him at $365\frac{5}{19}$ days, or 365 days 6 hours 16 minutes, although he probably knew that there was no authority for the odd 16 minutes. It follows that if, as is probable, πρωίας refers to a time later than 5.28 a.m., the third summer solstice would be reckoned for a time later than 6 p.m., probably not much later. This renders it probable that the 2nd year was assigned 366 days, and, if so, the 6th, 10th, 14th, and 18th.

It remains to be seen which months were intercalary. The only direct evidence on this subject is to be found in astronomical observations dated by months which appear to be Metonic. We have three such in Ptolemy, *Meg. Syn.*, **4**, 11 (p. 340 ff., ed. Heiberg). These are lunar eclipses, cited by Ptolemy from Hipparchus, who stated that they were observed at Babylon and belonged to a collection of observations brought from Babylon. As we have them, the dates are expressed by Attic archons and Attic months. Now it is out of the question that Hipparchus or any other astronomer should have had the necessary information to convert Babylonian months in distant years into months of the Attic civil calendar, or that succeeding astronomers could have been sure of the dates if they had been so converted. The Babylonian months must have been converted into Greek astronomical months—that is, into months of the calendar used by astronomers to express dates for the time of the observation, and, as the observations fall between the beginnings of the Metonic and Callippic cycles, the months named by Ptolemy must be Metonic. The fact that the initial day of the Metonic cycle could be converted into an Egyptian date proves that a continuous record of time was kept by means of that cycle. For solstitial and equinoctial observations at the beginnings of Metonic cycles compare those of Aristarchus in 280 B.C., Hipparchus in 147 B.C., the Miletus fragments in 109 B.C., and Ptolemy in 139-140 A.D. The value of these depends on the maintenance of an exact record of time going back to and conducted along the Metonic cycle. Hipparchus converted the dates of these Babylonian lunar eclipses into the Egyptian calendar, and we thus know that December 23 in 383 B.C. fell in Poseideon, June 18 in 382 B.C. in Scirophorion, and December 12 in 382 B.C. in the former Poseideon. The reference to "the former Poseideon" proves that the intercalary month was a second Poseideon, as in the Attic civil calendar. The two first of these dates fall in the 50th year from the beginning of the first Metonic cycle—that is, in the 12th year of the third cycle. The dates prove that (1) there must have been 606 months in the first 49 years, which means that there were 136 in the first 11 years of the third cycle, or, in other words, that there were four intercalations in those 11 years, and (2) the 12th year cannot have contained an intercalation, but the 13th must have. That is all that we can infer from this passage, and there is no other passage that gives us Metonic dates.

We know that the calendar of Meton and Euctemon was accompanied by a parapegma. This being so, the solar year doubtless formed the framework, and it is probable that the lunar months were shown by means of pegs as in the calendar of Miletus. In that case it seems natural to suppose that the first new month in each Metonic calendar year was regarded as Hecatombæon, as in the ideal calendar of Plato,

Laws, 767 C. This theory agrees exactly with the information obtained in the last paragraph from recorded dates. Assuming this theory to be sound, I construct the following table of the initial dates of each year in the first Metonic cycle. It will be understood that each day, month, and year was supposed to begin at the sunset preceding the date specified in my table.

Number of Year.	Days elapsed before New Year's Day.	Metonic Lunar Date of New Year's Day.	Julian Date of New Year's Day.
I.	...	Scir. 13	B.C. 432 June 27
II.	370 – 5	,, 23	431 ,, 27
III.	742 – 11	,, 5	430 ,, 28
IV.	1113 – 17	,, 16	429 ,, 27
V.	1484 – 23	,, 27	428 ,, 27
VI.	1854 – 28	,, 7	427 ,, 27
VII.	2226 – 34	,, 19	426 ,, 28
VIII.	2597 – 40	,, 30	425 ,, 27
IX.	2968 – 46	,, 11	424 ,, 27
X.	3339 – 52	,, 22	423 ,, 27
XI.	3710 – 57	,, 3	422 ,, 28
XII.	4081 – 63	,, 14	421 ,, 27
XIII.	4452 – 69	,, 25	420 ,, 27
XIV.	4823 – 75	,, 6	419 ,, 27
XV.	5195 – 81	,, 18	418 ,, 28
XVI.	5565 – 86	, 28	417 ,, 27
XVII.	5936 – 92	,, 9	416 ,, 27
XVIII.	6307 – 98	,, 20	415 ,, 27
XIX.	6679 – 104	,, 2	414 ,, 28
I.	7050 – 110	,, 13	413 ,, 27

From this table it would appear that the years in which intercalations were made were the 2nd, 5th, 8th, 10th, 13th, 16th, and 18th of the cycle. It will be observed that in the first twenty-five years the same arrangement of intercalations might have been found in a calendar regulated by an octaeteris. This illustrates the difficulty that exists in the way of determining by concrete examples what rule, if any, governed a calendar.

It might be objected to my restoration of the Metonic cycle that the omission of every 64th day, irrespective of its place in the month, might lead to the omission of the proper days for important festivals, to which I reply that in all probability Meton never expected his calendar to be used for other than astronomical and meteorological purposes, though he probably knew that it would provide a standard by which the errors of the civil calendar could be measured.

For the calendar of Callippus we have a description in Geminus and

several exact dates in Ptolemy. Geminus, 8, 57 ff., gives the following information: "Now in this (*i.e.* the Metonic) cycle the months seem to have been correctly defined and the intercalary months distributed in accordance with the phenomena, but the length of the year has not been defined in accordance with the phenomena. For the length of the year, as a result of many years' observations, is agreed to be $365\frac{1}{4}$ days, while the year resulting from the 19-year period is of $365\frac{5}{19}$ days. And these exceed $365\frac{1}{4}$ days by the 76th part of a day. Accordingly Callippus and his school corrected this excessive fraction of a day and constructed the 76-year period, consisting of four 19-year periods, and containing 940 months, of which 28 are intercalary, and 27,759 days. They adopted the same arrangement of intercalary months. And this cycle appears to agree best of all with the phenomena."

The only exact information here is on the length of the cycle and the statement that the arrangement of intercalary months was the same as in the Metonic cycle. In what sense this is true will be discussed later.

While dates belonging to the Metonic calendar are, as has been seen, expressed in years of Athenian archons, the Callippic cycle was used as a means of numbering years. The cycles, as well as the years of each cycle, were numbered, and we have a large number of such dates preserved to us in Ptolemy. Four dates cited from Timocharis, belonging to the first cycle, give the Callippic date for year, month, and day. A summer solstice in the same cycle cited from Aristarchus through the medium of Hipparchus gives the Callippic year, but no precise day, merely the end of the year. Numerous dates belonging to the second and third cycles are cited from Hipparchus, referring to his own observations or older observations used by him. Here we regularly have the Callippic year, but either no exact day, or, more often, the month and day of the Egyptian calendar.* The last date expressed in Callippic months and days falls in 283 B.C. The last date expressed in Callippic years falls in 128 B.C. With two exceptions, in which either Ptolemy or his copyists must have erred, all these dates point to the summer solstice of 330 B.C. as the initial date of the calendar. For more precise information we must turn to the four exact dates cited from Timocharis. The Julian equivalents of these dates are certain. Ptolemy gives the Egyptian equivalents, which have all been confirmed by astronomical computation of the observations.

But, before proceeding to restore the calendar from these observations, it may be well to attempt to gain by other means a knowledge of Callippus's date for the summer solstice, with which his cycle began. We have two sources for the intervals between the different solstices as determined by Callippus. One of these, the Εὐδόξου τέχνη, definitely attributes the following intervals to Callippus:—

Summer solstice to autumnal equinox . .	92 days
Autumnal equinox to winter solstice . .	89 ,,
Winter solstice to vernal equinox . . .	90 ,,

* See the index to Manitius's translation of Ptolemy's *Meg. Syn.* (*Handbuch der Astronomie*). The list given by Ginzel, *Handbuch*, 2, 410, is far from complete.

The calendar of Geminus cites Callippus as placing the year-points on the first day of the passage of the Sun through the respective signs as shown in that calendar, from which it is inferred that the calendar was intended to be based on Callippus. We cannot, however, be sure that the author did follow Callippus consistently. The intervals between solstices and equinoxes in that calendar work out as follows:—

Summer solstice to autumnal equinox . .	92 days
Autumnal equinox to winter solstice . .	89 „
Winter solstice to vernal equinox . . .	89 „
Vernal equinox to summer solstice . . .	95 „

I find by computation the following dates for the solstices and equinoxes for the year 330–29 B.C. I have taken the secular acceleration of the Sun's (or Earth's) motion from my own researches, the position and mean motion of the Sun in 1800 from Newcomb's Tables of the Sun, and the longitude of the solar perigee from the same source, and have applied the resultant corrections to the times of the respective year-points as computed from Schram's *Zodiakaltafeln* (*Kalendariographische und Chronologische Tafeln*, 1908). The dates are expressed in Athens mean time. I thus get:

Summer solstice . .	330 B.C.,	June 28, 2.5 a.m.
Autumnal equinox . .	330	Sept. 28, 6.35 a.m.
Winter solstice . .	330	Dec. 25, 8.27 p.m.
Vernal equinox . .	329	March 25, 5.40 a.m.

It will be observed that the intervals are:

Summer solstice to autumnal equinox .	$92^{d}\ 4^{h}\ 30^{m}$
Autumnal equinox to winter solstice . .	88 13 52
Winter solstice to vernal equinox . .	90 9 13

The intervals attributed to Callippus in the Εὐδόξου τέχνη are, therefore, correct to the nearest day, which is the degree of accuracy aimed at. Now it is easier to suppose that Callippus had deduced the correct dates for all the year-points than that he made an identical error for all four. If we assume that Callippus reckoned the day from mean sunset, we have 330 B.C. June 27–8 as the date of his initial summer solstice.

The date at which each Callippic month began can be restored from the recorded dates, if we have a theory as to the length of each Callippic month. In default of other information on this subject, I assume that, like Meton, he attributed 30 days to each month and then dropped every 64th day. In the Metonic cycle there were 10 days left over after the last of the 110 days had been eliminated. In the Callippic cycle there are 441 days to eliminate, and if the first 440 are made at regular intervals, we shall have to assume that the 441st day eliminated was the 40th day after the 440th, and was actually the last day of the cycle. The question is not important, since the last exact Callippic date belongs to the 48th year of the cycle, nearly 29 years before its end. In the

following table it is assumed that every 64th day was eliminated. I now analyse the four dates cited by Ptolemy from Timocharis: all the observations were made at night. Hence the double Julian date.

Year of Cycle.	Callippic Date.	Julian Date.	Number of Days from Initial Solstice.	Number of eliminated Days before Observation.	Resultant Date of Hec. 1 in First Year of Cycle.
36	Poseideon 25	B.C. 295 Dec. 20–1	12,960	205	B.C. 330 June 28–9
36	Elaphebolion 15	294 Mar. 9–10	13,039	206	330 ,, 28–9
47	Anthesterion 8	283 Jan. 29–30	17,018	270	330 ,, 28–9
48	Pyanepsion 6φθ = Py. 25	283 Nov. 8–9	17,301	274	330 ,, 28–9

It will thus be seen that all four dates point to the same date for Hec. 1 in the first year of the cycle. This, so far as it goes, confirms the view that Callippus, like Meton, eliminated every 64th day. The dates also tell us something about intercalation. In B.C. 283 we have Pyanepsion falling nine months later than Anthesterion. The normal interval in the Attic calendar was eight months. It follows, therefore, that there must have been an intercalary month in this interval. If so, intercalation cannot have been made by doubling Poseideon, and, since the Greek calendars known to us nearly all doubled either the 6th or the 12th month when an intercalation was necessary, the inference is that in the Callippic cycle intercalation was effected by doubling the 12th month, Scirophorion.

The first night of the Callippic Hecatombæon in 330 B.C. has been found to be June 28–9. I find by computation that the mean new moon that month fell on June 28 at 2.59 p.m., Athens mean time. It would appear, therefore, that this month was made to begin at the sunset following mean new moon rather than at the sunset preceding mean new moon, as in the Metonic calendar. I have found that the same rule obtained in the lunar calendar employed in the Aramaic papyri from Elephantine (*Monthly Notices of the Royal Astronomical Society*, **69** (1908), 12–20; **69** (1909), 446–448; *Errata*, 470, 542; **72** (1911), 661–663; *Journal of Theological Studies*, **14** (1913), 570–575). If this is so, it is reasonable to suppose that the solar year was made to begin at the sunset following the actual solstice, and the initial day of the cycle would then coincide with the initial day of the first month, June 28–9, 330 B.C. If the Callippic year was reckoned from the sunset following the summer solstice, this will help to explain the statements that Aristarchus and Hipparchus observed the solstice at the *end* of the 50th year of the first cycle and the 43rd year of the third cycle respectively. But we also know that Aristarchus's observation was made too early, and led Hipparchus to deduce an excessive length for the tropical year, while by the time of Hipparchus the actual year-points would be falling about 1½ day before the time for which Callippus had computed them.

The theory that the months were intended to begin at the sunset following mean new moon is confirmed by the following table, computed for the first year of the cycle :—

Number of Month.	Days elapsed before beginning of Month.	Julian Date of First Day.	Mean New Moon.
1	...	June 29	June 28, 2.59 p.m.
2	30	July 29	July 28, 3.43 a.m.
3	60	Aug. 28	Aug. 26, 4.27 p.m.
4	90–1	Sept. 26	Sept. 25, 5.11 a.m.
5	120–1	Oct. 26	Oct. 24, 5.55 p.m.
6	150–2	Nov. 24	Nov. 23, 6.39 a.m.
7	180–2	Dec. 24	Dec. 22, 7.23 p.m.
8	210–3	Jan. 22	Jan. 21, 8.8 a.m.
9	240–3	Feb. 21	Feb. 19, 8.52 p.m.
10	270–4	Mar. 21	Mar. 20, 9.36 a.m.
11	300–4	Apr. 20	Apr. 18, 10.20 p.m.
12	330–5	May 19	May 18, 11.4 a.m.
13	360–5	June 18	June 16, 11.48 p.m.

Here it will be observed that the 3rd and 5th months appear to be reckoned from the second mean sunset after mean new moon, the others from the sunset immediately following mean new moon. The 5th month is of no importance, for it is certain that Callippus could not compute the time of mean new moon to 5 minutes. The remaining exception could be removed if the 64-day period were made to begin from 4 to 11 days before the first day of the cycle, but it is more likely that Callippus did not aim at such precision.

If, as supposed, it was Callippus's design to make each year of his cycle begin at the mean sunset following the summer solstice, the 3rd year should receive 366 days, and so should each year whose number when divided by 4 left 3 as a remainder. As the first year of his cycle was the eighth of a Metonic cycle, and, in fact, on the present suppositions began on the same day as the corresponding Metonic year, the third Callippic year should be the tenth of a Metonic cycle, which, according to the reconstruction attempted in this discussion, should also have contained 366 days, and the initial days of the year in the two systems ought to agree as far as the second year of the next Metonic cycle, which would be the fourteenth of the first Callippic cycle.

Returning to our analysis of the four recorded Callippic dates, we find that Poseideon of year 36 must have been the 439th month of the cycle, from which it follows that the first 35 years must have contained 13 intercalary months, and that Hec. 1 of that year must have fallen 12,990 − 202 or 12,788 days after the initial day of the cycle, *i.e.* on July 3 Julian; Elaphebolion of the same year must have been the 442nd month of the cycle, from which it follows again that the first 35 years must have contained 13 intercalary months, and that Hec. 1 of that year must have been July 3 Julian; Anthesterion of the 47th year must have been the 577th month of the cycle, from which it follows that the first 46 years must have contained 17 intercalary months, and that Hec. 1 of that year must have been July 1 Julian; while Pyanepsion of the 48th year must have been the 586th month of the cycle, from which it follows that the first 47 years must have contained

18 intercalary months, and that Hec. 1 of that year must have been July 20 Julian. It will, therefore, be seen that in each of these years the rule, believed to be Metonic, is preserved that the first new month in each year is regarded as Hecatombæon. As the Callippic cycle begins with the same New Year's Day as the Metonic, but places Hec. 1 one day earlier, the intercalary months ought to fall in the same actual years in both calendars, being six months later in the Callippic than in the Metonic system. The intercalations would, therefore, be made in years—

1, 3, 6, 9, 11, 14, 17,
20, 22, 25, 28, 30, 33, 36,
39, 41, 44, 47, 49, 52, 55,
58, 60, 63, 66, 68, 71, 74—

of the cycle. The actual dates for the beginnings of years and months would gradually fall later and later in the Metonic than in the Callippic cycle, at the rate of 1 day in 76 years.

The identification of particular months of both calendars with particular lunations in the restoration attempted in this paper is in agreement with the restorations proposed by Scaliger, Müller,* and August Mommsen.† But the restorations differ both in principle and in details. Mommsen's, for example, makes each Metonic cycle begin one year earlier than I, in common with others, have supposed. I may note, for what it is worth, that the eclipse of the Sun recorded by Thucydides (2, 28) in the second year of the first Metonic cycle, which, he says, fell *νουμηνίᾳ κατὰ σελήνην* would actually fall on the *νουμηνία* or first day of the month in Meton's calendar according to Scaliger, Dodwell, Müller, and myself, but not according to most modern scholars.

University Observatory, Oxford:
1924 *February* 21.

On a Theorem of Sir Isaac Newton. By A. N. Kriloff, Member of the Russian Academy of Sciences, Petrograd.

(*Communicated by Frank Robbins.*)

§ 1. For the determination of parabolic orbits of comets, the well-known relation between the time of description of an arc of the orbit, the two radii vectores drawn to the extremities of this arc, and the chord is of fundamental importance.

This relation is called in all modern treatises on theoretical astronomy, even in English ones, Lambert's or Euler-Lambert's theorem, although Lagrange in his *Mécanique Analytique* pointed out that it is only the analytical expression of the Xth Lemma in Book III. of Newton's *Principia.*

* Pauly's *Real-Encyclopädie der Alterthumswissenschaft,* 2 Auflage, Band i, heft 1 (1864), 1038–1060. † *Chronologie* (1883).

It is most remarkable that even in such a classical treatise as Bauschinger's *Bahnbestimmung* we find on p. 388 the following sentence: "Newton's method was a graphical one and could be used in practice, . . . what was wanted to make it perfect was the theorem of Euler-Lambert, instead of which Newton used an approximate relation between the time, the radii vectores, and the chord." What Bauschinger calls "an approximate relation" is just the same relation which constitutes the Euler-Lambert theorem, but geometrically expressed.

Lagrange gives only a verbal sketch of a proof of his assertion, leaving it to the reader to supply the necessary algebra. This may be the reason why his remark has been overlooked.

The aim of this note is to show that the required algebraical transformations can be performed in a very simple manner.

§ 2. Newton's lemma is:

If on Sμ *produced lengths* μN *and* μP *be set off so that* $\mu N = \frac{1}{3} I\mu$ *and*

$$SP : SN = SN : S\mu,$$

SP *will be the distance from the sun* S *at which the comet has such a velocity that it would describe (in uniform motion) a distance equal to the chord* AC *in the same time in which it describes the arc* AμC *of its parabolic trajectory.*

[S is the focus of a parabola, μ is the point on the arc AC such that the tangent μT is parallel to the chord AC, and I is the mid point of the chord AC.]

Using modern notation, Newton's proof can be expressed as follows: If the comet were to continue its motion from μ along the tangent μT during the same time in which it describes the arc AμC, the length μT would be such that the area SμT = SAμC.

But we have, if Sμ and AC meet in M at an angle α,

$$SA\mu C = ASC + AC\mu = \tfrac{1}{2} AC \,.\, SM \,.\, \sin\alpha + \tfrac{2}{3} I\mu \,.\, AC \,.\, \sin\alpha$$
$$= \tfrac{1}{2} AC(SM + \tfrac{1}{3} I\mu) \,.\, \sin\alpha = \tfrac{1}{2} AC \,.\, SN \,.\, \sin\alpha,$$

and

$$S\mu T = \tfrac{1}{2} S\mu \,.\, \mu T \,.\, \sin\alpha,$$

hence

$$\mu T : AC = SN : S\mu.$$

In the parabolic motion the squares of the velocities are in the inverse ratio of the distances from the sun, hence with an obvious notation,

$$V_\mu : V_P = SP^{\frac{1}{2}} : S\mu^{\frac{1}{2}},$$

but by construction

$$SP = \frac{SN^2}{S\mu},$$

hence

$$\frac{SP}{S\mu} = \frac{SN^2}{S\mu^2},$$

therefore

$$V_\mu : V_P = SN : S\mu = \mu T : AC,$$

hence

$$\frac{\mu T}{V_\mu} = \frac{AC}{V_P} = \tau,$$

where τ is the time of describing the arc $A\mu C$.

§ 3. According to this lemma and a previous one, the time τ can be expressed in terms of the chord AC and the radius $S\mu$, drawn to the vertex μ of the arc $A\mu C$. Indeed the previous lemma IX. of the same book expresses the relations :

$$I\mu = M\mu = \frac{AI^2}{4S\mu},$$

"patet ex conicis," says Newton, in lemma XIII. of the first book, where this same property of the parabola is mentioned. Using the following notation :

$$S\mu = \zeta; \quad SA = r_1; \quad SC = r_2; \quad AC = c; \quad I\mu = \gamma,$$

we have

$$I\mu = M\mu = \frac{AI^2}{4S\mu} = \gamma = \frac{c^2}{16\zeta},$$

$$SN = S\mu + \tfrac{1}{3}I\mu = \zeta + \tfrac{1}{3}\gamma = \zeta + \frac{c^2}{48\zeta},$$

$$SP = \frac{SN^2}{S\mu} = \frac{(\zeta + \frac{1}{3}\gamma)^2}{\zeta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tau = \frac{c}{V_P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By a property of parabolic motion

$$V_P = \frac{\kappa\sqrt{2}}{SP^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where κ is a constant for the solar system (afterwards called the Gaussian constant) of which Newton gives the numerical value and the analytical expression. From the equations (1), (2), (3), we have

$$\kappa\sqrt{2} \,.\, \tau = c \,.\, SP^{\frac{1}{2}} = \frac{c}{\sqrt{\zeta}}\left(\zeta + \frac{c^2}{48\zeta}\right) \quad . \quad . \quad . \quad (4)$$

§ 4. If we introduce into the analytical expression (4) of Newton's lemma instead of the radius ζ, the radii r_1 and r_2 of the extremities of the arc AC, we obtain the Euler-Lambert theorem.

Denoting by p the *latus rectum* of the parabola and by x_0, x_1, x_2 the abscissæ of the points μ, A, C respectively, we have

$$\zeta = \frac{p}{2} + x_0,$$

$$r_1 = \frac{p}{2} + x_1,$$

$$r_2 = \frac{p}{2} + x_2,$$

hence

$$\frac{r_1+r_2}{2}-\zeta=\frac{x_1+x_2}{2}-x_0=\mathrm{I}\mu=\frac{c^2}{16\zeta},$$

and we obtain the equation :

$$\zeta^2-\frac{r_1+r_2}{2}\zeta+\tfrac{1}{16}c^2=0 \quad . \quad . \quad . \quad . \quad (5)$$

which gives

$$\zeta=\frac{r_1+r_2}{4}+\tfrac{1}{4}\sqrt{(r_1+r_2+c)(r_1+r_2-c)}.$$

Hence, if we put

$$r_1+r_2+c=g,$$
$$r_1+r_2-c=f,$$

equation (5) becomes

$$\zeta^2-\frac{g+f}{4}\zeta+\tfrac{1}{64}(g-f)^2=0 \quad . \quad . \quad . \quad . \quad (5')$$

which gives

$$\zeta=\tfrac{1}{8}[g+f+2\sqrt{gf}]=\tfrac{1}{8}(\sqrt{g}+\sqrt{f})^2$$

and

$$\frac{c}{\sqrt{\zeta}}=\frac{2\,.\,\sqrt{2}(g-f)}{2(\sqrt{g}+\sqrt{f})}=\sqrt{2}(\sqrt{g}-\sqrt{f}).$$

By equation (5) we have :

$$\frac{c^2}{48\zeta}=\tfrac{1}{6}(r_1+r_2)-\tfrac{1}{3}\zeta,$$

hence

$$\zeta+\frac{1}{48}\frac{c^2}{\zeta}=\tfrac{2}{3}\zeta+\tfrac{1}{6}(r_1+r_2)=\tfrac{1}{6}[4\zeta+(r_1+r_2)]$$
$$=\tfrac{1}{6}[\tfrac{1}{2}(\sqrt{g}+\sqrt{f})^2+\tfrac{1}{2}(g+f)]$$
$$=\tfrac{1}{6}[g+\sqrt{gf}+f].$$

Substituting in (4) we obtain

$$\kappa\sqrt{2}\tau=\frac{\sqrt{2}}{6}(\sqrt{g}-\sqrt{f})(g+\sqrt{gf}+f)=\frac{\sqrt{2}}{6}[(\sqrt{g})^3-(\sqrt{f})^3]$$

or

$$6\kappa\tau=(r_1+r_2-c)^{\frac{3}{2}}-(r_1+r_2-c)^{\frac{3}{2}} \quad . \quad . \quad . \quad (\mathrm{A})$$

This expression constitutes the so-called Euler-Lambert theorem.

The above elementary and straightforward calculation shows clearly that this theorem is nothing else than the analytical expression of Newton's X$^{\text{th}}$ lemma.

We have tacitly assumed, as Newton does, that the angle between the two extreme radii vectores is less than 180° ; if this angle were more than 180° the sign plus would be obtained in formula (A), as can easily be proved.

A lot of different proofs of the formula (A) can be found in modern treatises on theoretical astronomy, but Newton's remains unsurpassed in its wonderful dynamical insight.

Observations of Novæ, 1923. By W. H. Steavenson.

All the observations here recorded were made with the 6-inch Wray refractor. For Nova Aquilæ (1918) and Nova Cygni (1920) a power of 120 was used, and for the other and fainter Novæ powers between 170 and 310.

Nova Aquilæ (1918).

Observed on 85 nights between 1923 April 26 and November 16. The fading of this Nova, which ceased for a considerable time during 1922 (*M.N.*, **83**, 161), was resumed in 1923, amounting to a further $0^{m}\cdot1 \pm$ during the period covered by the observations. In addition, about $0^{m}\cdot1$ appears to have been lost in the interval between the two periods, so that the Nova was, in November 1923, about $0^{m}\cdot2$ fainter than in the summer of 1922.

The individual estimates for 1923 are not given, as they differ but little from night to night, but the observations have been grouped so as to give the mean brightness at intervals of 10 days. The groups overlap, the value adopted for each date covering also the preceding and following 10 days, or 21 days in all. It will be noted that there were halts with slight recovery about June 8 and October 1. Otherwise the fall has been sensibly rectilinear, and the individual observations show no evidence of any short-period oscillation of an amplitude as great as $0^{m}\cdot1$.

Date.	Mag.	No. of Nights.	Date.	Mag.	No. of Nights.
1923 May 4	10·10	5	1923 Aug. 12	10·18	12
14	10·13	8	22	10·18	7
24	10·14	6	Sept. 1	10·19	10
June 3	10·17	8	11	10·21	12
13	10·17	8	21	10·22	8
23	10·13	5	Oct. 1	10·25	4
July 3	10·15	10	11	10·21	8
13	10·16	13	21	10·19	9
23	10·17	13	31	10·19	8
Aug. 2	10·17	13	Nov. 10	10·23	6

Nova Cygni (1920).

Observed on 85 nights between 1923 April 26 and December 24. The Nova continues to fade, though less rapidly than in 1922. In the latter year, during the period covered by the observations, the mean daily fall amounted to $^{m}\cdot0038$, and was as great as $^{m}\cdot0075$ in May and June. In 1923 the rate was almost steady, and was only $^{m}\cdot0017$ per

day. There is no certain evidence of short-period oscillations of an amount greater than the probable error of the observations. The Nova has always been a difficult object to observe accurately, and is still relatively prominent to averted vision, though less so than in 1921 and 1922. The following values of the magnitude have been deduced in the same way as those of Nova Aquilæ. There is some suggestion of a slight halting at intervals of about 50 days, but on the whole the decline has been very steady.

Date.	Mag.	No. of Nights.	Date.	Mag.	No. of Nights.
1923 May 4	11·46	5	1923 Aug. 22	11·64	7
14	11·45	8	Sept. 1	11·67	10
24	11·49	7	11	11·69	11
June 3	11·49	8	21	11·69	6
13	11·51	9	Oct. 1	...	...
23	11·55	7	11	11·71	7
July 3	11·55	11	21	11·71	12
13	11·56	13	31	11·69	8
23	11·59	13	Nov. 10	11·72	3
Aug. 2	11·59	13	* * * * * * * *		
12	11·59	12	Dec. 30	11·83	3

Nova Ophiuchi (1848).

Observed on 34 nights between 1923 April 26 and September 6. The observations show a continuation of the variability observed in previous years, but the range for 1923 ($0^m{\cdot}5 \pm$) was less than in 1922 ($0^m{\cdot}75 \pm$). There is a strong suggestion of a period of about 49 days, which would fit also the observations made in 1922, for which a period of $50^d \pm$ was provisionally suggested (*M.N.*, 83, 161). The persistence of this periodicity makes it probable that the variability of the Nova is not entirely irregular. It would seem very desirable that a more complete series of observations should be undertaken in a good climate with a large instrument.

As observed in the 6-inch, the image of the Nova generally gives the impression of being not quite stellar in character, and appears unduly large to averted vision as compared with the nearest comparison star. It also appears relatively faint in moonlight or twilight, as do Nova Cygni (1920), Nova Aquilæ (1918), and Nova Persei (1901), which probably indicates that it is closely surrounded by nebulosity, and perhaps has also a negative colour index.

[Table

Date.	Mag.	Date.	Mag.	Date.	Mag.
1923 Apr. 26	11·9±	1923 July 5	12·3±	1923 Aug. 6	12·3
May 9	12·59	8	12·45	7	12·1
11	12·56	11	12·35	8	12·2
15	12·45	13	12·4	11	12·2±
16	12·35	20	12·25	15	12·1
17	12·35	21	12·4	18	12·1
21	12·4±	31	12·2	21	12·2
June 7	12·15	Aug. 2	12·15	28	12·25
10	12·05	3	12·1	30	12·35
22	12·3	4	12·15	31	12·4
July 2	12·2	5	12·15	Sept. 6	12·5
4	12·4				

Nova Cygni (1876).

Observed on 24 nights between 1923 May 11 and September 18. The faintness of this object makes it impossible to observe with accuracy with an aperture as small as 6 inches, and it is only on very clear nights that the Nova is visible at all. The following values of the magnitude are probably subject to some uncertainty, but they are sufficiently exact to indicate whether the Nova was sensibly above or below its average brightness on the dates given. There is no doubt of the reality of the variations, but a larger instrument is needed for a satisfactory study of them.

Date.	Mag.	Date.	Mag.	Date.	Mag.
1923 May 11	14·57	1923 Aug. 5	14·86	1923 Sept. 5	14·61
16	14·76	6	14·91	6	14·71
July 20	14·62	7	14·91	8	14·71
21	14·81	8	14·86	11	14·81
24	14·81	11	14·86	15	14·61
Aug. 2	14·66	15	14·81	16	14·66
3	14·81	18	14·81	17	14·56
4	14·86	21	14·76	18	14·61

Nova Lacertæ (1910).

Observed on 20 nights between 1923 May 11 and September 18. As in previous years, the Nova appeared to be sensibly constant in its light. All the estimates lie between $14^{m}\cdot0$ and $14^{m}\cdot1$, except those of July 21 and August 4 and 5, which were $14^{m}\cdot2$ and $13^{m}\cdot9$ respectively. It is doubtful, however, whether any real change took place.

Observations of Occultations, 1923. By W. H. Steavenson.

The following occultations (all disappearances at the dark limb) were observed with the 6-inch Wray refractor, with a power of 150. The times were taken from the Cooke Mean Time clock, used in conjunction with a microphone. The error of the clock, determined by Eiffel Tower signals, was known in each case within about $0^{s}\cdot1$, but the probable error of the observations is likely to be of the order of $0^{s}\cdot3$. The longitude of the observatory is $0^{h}\ 0^{m}\ 23^{s}\cdot10$ W., and the latitude 51° 26′ 9″·9 N. (Ordnance Survey). The height above sea-level is 125 feet.

Date.	Star.	B.D. Mag.	G.M.T. of Disappearance.
			h m s
1923 Jan. 27	B.D. +15°·637 (264B. Tauri)	5·0	9 25 18·5
27	+15·640 (W.Z.C. 305)	7·5	9 36 21·5
27	+16·629 (α Tauri)	1·1	12 35 48·9
Feb. 23	*+14·6557 (179B. Tauri)	6·5	7 13 41·7
May 24	+ 4·2461 (W.Z.C. 745)	7·3	10 13 45·95
24	+ 3·2502 (83 Leonis)	7·5	10 34 30·9
24	+ 3·2503	8·0	10 35 1·9
24	+ 3·2504 (τ Leonis)	5·0	11 10 30·35
24	+ 3·2505	8·2	11 11 13·05

* This is the double star Σ495. The *comes* (220°·7, 3″·6, $8^{m}\cdot8$) disappeared about four seconds earlier, but the exact time was not noted.